Chapter 14: The principal-agent problem

- Hidden actions (Moral hazard) (§14.B)
- Hidden information (§14.C)

§14.B Hidden actions

Principal-Agent model The owner of a firm (the principal) hires a manager (the agent).

The effort level of the manager The effort is not observable nor verifiable. e_H: high effort, e_L: low effort.

§14.B Hidden actions

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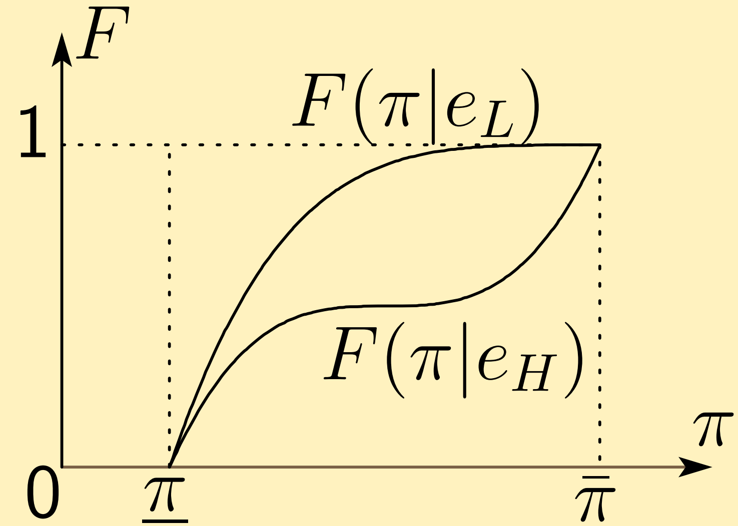
The profit of the firm π This is observable and verifiable. This depends on the density function $f(\pi|e)$ on $[\underline{\pi}, \bar{\pi}]$ given e.

$$\forall e,\ \forall \pi \in [\underline{\pi}, \bar{\pi}],\ f(\pi|e) > 0.$$

$$\forall \pi \in (\underline{\pi}, \bar{\pi}),\ F(\pi|e_H) < F(\pi|e_L).$$



Hidden actions (Moral hazard)

The owner's profit $\pi - w$, where w is the wage.

The manager's net utility $u(w, e) = v(w) - g(e)$,

where $v' > 0$, $v'' < 0$, $g(e_H) > g(e_L)$.

$\bar{u}$: The manager's reservation utility.

Hidden actions (Moral hazard)

Effort is observable and verifiable (benchmark)

$$\begin{aligned} \max_{e \in \{e_L, e_H\}, w(\pi)} & \int (\pi - w(\pi)) f(\pi|e) d\pi \\ s.t. & \int v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \bar{u}. \end{aligned} \tag{1}$$

1st, find optimal w for each e; 2nd, find the optimal e.

The first stage This is equivalent to

$$\begin{aligned} \min_{w(\pi)} & \int w(\pi) f(\pi|e) d\pi \\ s.t. & \int v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \bar{u}. \end{aligned} \tag{2}$$

Hidden actions (Moral hazard)

The first stage $\min_{w(\pi)} \int w(\pi) f(\pi|e) d\pi$

$$s.t. \int v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \bar{u}.$$

F.O.C. $w(\pi)$ at each level of $\pi \in [\underline{\pi}, \bar{\pi}]$ must satisfy

$$-f(\pi|e) + \gamma v'(w(\pi)) f(\pi|e) = 0, \ or \ \frac{1}{v'(w(\pi))} = \gamma, \quad (3)$$

where γ is the Lagrange multiplier. Thus, regardless of π, $w(\pi)$ is constant (w_e^*). The risk-neutral owner fully insures the risk-averse manager.

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where γ is the Lagrange multiplier. Thus, regardless of π, $w(\pi)$ is constant (w_e^*). The risk-neutral owner fully insures the risk-averse manager.

The optimal wage Since the constraint is binding,

$$\int v(w_e^*) f(\pi|e) d\pi - g(e) = \bar{u} \ \rightarrow \ w_e^* = v^{-1}(g(e) + \bar{u}).$$

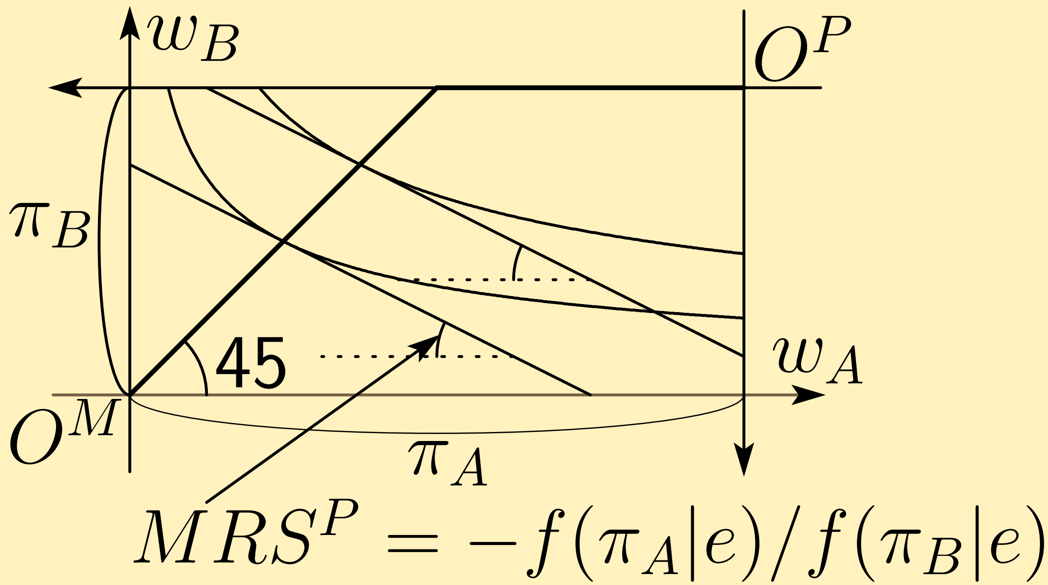
Full insurance Suppose that there are only two events π_A and π_B.

$$U^P = -\left[f(\pi_A|e)w_A + f(\pi_B|e)w_B\right], \tag{4}$$

$$MRS^P = -f(\pi_A|e)/f(\pi_B|e). \tag{5}$$

$$U^M = f(\pi_A|e)v(w_A) + f(\pi_B|e)v(w_B), \tag{6}$$

$$MRS^M = -[f(\pi_A|e)v'(w_A)]/[f(\pi_B|e)v'(w_B)]. \tag{7}$$



Hidden actions (Moral hazard)

The second stage Given $w_e^* = v^{-1}(g(e) + \bar{u})$ in stage 1,

$$\max_e \int (\pi - v^{-1}(g(e) + \bar{u}))f(\pi|e)d\pi$$
$$\to \max_e \int \pi f(\pi|e)d\pi - v^{-1}(g(e) + \bar{u}). \tag{8}$$

Hidden actions (Moral hazard)

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$$\to \max_e \int \pi f(\pi|e)d\pi - v^{-1}(g(e) + \bar{u}). \qquad (8)$$

Proposition 14.B.1 In the principal-agent model with observable (and verifiable) managerial effort, an optimal contract satisfies that the manager chooses the effort e^* that maximizes (8) and pays the manager a fixed wage $w^* = v^{-1}(g(e^*) + \bar{u})$.

Hidden actions (Moral hazard)

Example 1 $\pi \in [0,2]$, $\bar{u} = 0$, $v(w) = \sqrt{w}$, $v^{-1}(v) = v^2$, $g(e_H) = a \in (0,1)$, $g(e_L) = 0$, For $\pi \in [0,2]$,

$$f(\pi|e_L) = 1 - \frac{\pi}{2}, \quad E(\pi|e_L) = \int_0^2 \pi f(\pi|e_L) = \frac{2}{3},$$

$$f(\pi|e_H) = \frac{\pi}{2}, \quad E(\pi|e_H) = \int_0^2 \pi f(\pi|e_H) = \frac{4}{3}.$$

Hidden actions (Moral hazard)

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$$f(\pi|e_H) = \frac{\pi}{2}, \quad E(\pi|e_H) = \int_0^2 \pi f(\pi|e_H) = \frac{4}{3}.$$

The wage schedule is

$$w_L^* = v^{-1}(g(e_L) + \bar{u}) = 0, \quad w_H^* = v^{-1}(g(e_H) + \bar{u}) = a^2.$$

Hidden actions (Moral hazard)

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$$f(\pi|e_L) = 1 - \frac{\pi}{2}, \quad E(\pi|e_L) = \int_0^2 \pi f(\pi|e_L) = \frac{2}{3},$$

$$f(\pi|e_H) = \frac{\pi}{2}, \quad E(\pi|e_H) = \int_0^2 \pi f(\pi|e_H) = \frac{4}{3}.$$

The wage schedule is

$$w_L^* = v^{-1}(g(e_L) + \bar{u}) = 0, \quad w_H^* = v^{-1}(g(e_H) + \bar{u}) = a^2.$$

The condition that the owner implements e_L is

$$E(\pi|e_L) - w_L^* \geq E(\pi|e_H) - w_H^* \quad \Leftrightarrow \quad a \geq \sqrt{\frac{2}{3}}.$$

Hidden actions (Moral hazard)

Optimal contract: When the manager is risk-neutral (benchmark) When the effort is observable, (8) becomes as follows:

$$\max_{e} \int \pi f(\pi|e)d\pi - [g(e) + \bar{u}]. \tag{9}$$

Hidden actions (Moral hazard)

Optimal contract: When the manager is risk-neutral (benchmark) When the effort is observable, (8) becomes as follows:

$$\max_{e} \int \pi f(\pi|e)d\pi - [g(e) + \bar{u}]. \tag{9}$$

Proposition 14.B.2 In the principal-agent model with unobservable managerial effort and risk-neutral manager, an optimal contract generates the same effort choice and expected utilities for the manager and the owner as when effort is observable.

Hidden actions (Moral hazard)

Proof: The owner can never do better when effort is not observable than it is. Thus, if a contract gives the owner the same maximal payoff that he receives under full information, it is optimal.

Suppose that the owner offers $w(\cdot)$ where

$$w(\pi) = \pi - \alpha^*, \text{ } where \text{ } \alpha^* \equiv \int \pi f(\pi|e^*)d\pi - [\bar{u} + g(e^*)].$$

The manager's expected utility is

$$\int w(\pi)f(\pi|e)d\pi - g(e^*) = \int \pi f(\pi|e)d\pi - g(e) - \alpha^*.$$

Let e^* be the optimal effort in eq. (9). e^* also maximizes this expected utility.

Hidden actions (Moral hazard)

Proof (cont.) That is, the manager chooses e^* under the wage schedule $w(\cdot)$. The manager's expected utility is

$$\int \pi f(\pi|e^*)d\pi - g(e^*) - \alpha^* = \bar{u}.$$

He/She accepts the wage. Offering the wage schedule $w(\cdot)$, the owner obtains the expected payoff:

$$\int (\pi - w(\pi))f(\pi|e^*)d\pi = \int \alpha^* f(\pi|e^*)d\pi = \alpha^*.$$

This is the maximal profit when effort is observable.

Hidden actions (Moral hazard)

Effort is unobservable and manager is risk-averse

Stage 1 We find the optimal contract for e.

$$\min_{w(\pi)} \int w(\pi) f(\pi|e) d\pi \tag{10}$$

$$s.t.(i) \int v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \bar{u},$$

Individual Rationality (IR)

(ii) e solves

$$\max_{\hat{e} \in \{e_L, e_H\}} \int v(w(\pi)) f(\pi|\hat{e}) d\pi - g(\hat{e}).$$

Incentive Compatibility (IC)

Hidden actions (Moral hazard)

Case 1: Implementing e_L (ii) is

$$\int v(w(\pi))f(\pi|e_L)d\pi - g(e_L)$$
$$\geq \int v(w(\pi))f(\pi|e_H)d\pi - g(e_H).$$

We now consider the following wage offer

$$\bar{w}(\pi) = w_e^* = v^{-1}(\bar{u} + g(e_L)) \text{ for all } \pi.$$

This wage satisfies both (i) and (ii). In addition, the owner obtains the same profit as when effort is observable. This wage schedule optimally implements e_L even when effort is unobservable.

Hidden actions (Moral hazard)

Case 2: Implementing e_H (ii) is

$$\int v(w(\pi))f(\pi|e_H)d\pi - g(e_H)$$
$$\geq \int v(w(\pi))f(\pi|e_L)d\pi - g(e_L).$$

Solving the owner's optimization problem, we have

$$-f(\pi|e_H) + \gamma[v'(w(\pi))f(\pi|e_H)]$$
$$+ \mu[f(\pi|e_H) - f(\pi|e_L)]v'(w(\pi)) = 0.$$

where γ and μ are the multipliers on constraints (i) and (ii) respectively.

$$f(\pi|e_H) = v'(w(\pi))[\gamma f(\pi|e_H) + \mu[f(\pi|e_H) - f(\pi|e_L)]].$$

Case 2 (cont.)

$$
\begin{aligned}
f(\pi|e_H) =& v'(w(\pi))[\gamma f(\pi|e_H) + \mu[f(\pi|e_H) - f(\pi|e_L)]], \\
\frac{1}{v'(w(\pi))} =& \frac{1}{f(\pi|e_H)}[\gamma f(\pi|e_H) + \mu[f(\pi|e_H) - f(\pi|e_L)]], \\
=& \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right]. \qquad (11)
\end{aligned}
$$

Lemma 14.B.1 In any solution to (10) with $e = e_H$, $\gamma > 0$ and $\mu > 0$.

Proof Suppose that $\gamma = 0$. Since $F(\cdot|e_H)$ FOS dominates $F(\cdot|e_L)$, there is an open interval $\tilde{\Pi} \subset [\underline{\pi}, \bar{\pi}]$ such that $f(\pi|e_L)/f(\pi|e_H) > 1$ for all $\pi \in \tilde{\Pi}$.

Proof (cont.) By $\gamma = 0$ and $\mu \geq 0$, $\forall\ \pi \in \tilde{\Pi}$, (11) is

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)} \right] \leq 0,$$

that is, $v'(w(\pi)) \leq 0$. Since $v' > 0$, $\gamma > 0$.

Proof (cont.) By $\gamma = 0$ and $\mu \geq 0$, $\forall\ \pi \in \tilde{\Pi}$, (11) is

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right] \leq 0,$$

that is, $v'(w(\pi)) \leq 0$. Since $v' > 0$, $\gamma > 0$.

Suppose $\mu = 0$. Then, $1/v'(w(\pi)) = \gamma$, that is, $w(\pi)$ is constant. The manager chooses e_L rather than e_H (this violates (ii)). Hence, $\mu > 0$.

Proof (cont.) By $\gamma = 0$ and $\mu \geq 0$, $\forall\, \pi \in \tilde{\Pi}$, (11) is

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)} \right] \leq 0,$$

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Suppose $\mu = 0$. Then, $1/v'(w(\pi)) = \gamma$, that is, $w(\pi)$ is constant. The manager chooses e_L rather than e_H (this violates (ii)). Hence, $\mu > 0$.

Fact 14.B.1 Let $\hat{w}$ such that $1/v'(\hat{w}) = \gamma$. For all π

$$w(\pi) > \hat{w} \ \text{ if } \frac{f(\pi|e_L)}{f(\pi|e_H)} < 1, \quad w(\pi) < \hat{w} \ \text{ if } \frac{f(\pi|e_L)}{f(\pi|e_H)} > 1.$$

$w(\pi)$ is not necessarily monotonically increasing in π.

Example 2 $\pi \in [0,2]$, $g(e_H) = a \in (0,2)$, $g(e_L) = 0$, $\bar{u} = b > a$, $v(w) = \sqrt{w}$, $v^{-1}(v) = v^2$.

For $\pi \in [0,2]$,

$$f(\pi|e_L) = \begin{cases} 3/4 & \text{if } \pi \le 1, \\ 1/4 & \text{if } \pi > 1, \end{cases} \quad f(\pi|e_H) = \frac{1}{2},$$

$$v'(w) = 1/(2\sqrt{w}).\ L(\pi) \equiv \frac{f(\pi|e_L)}{f(\pi|e_H)} = \begin{cases} 3/2 & \text{if } \pi \le 1, \\ 1/2 & \text{if } \pi > 1. \end{cases}$$

Let $w(\cdot)$ be an optimal solution to (10) implementing $e = e_H$. (11) is

$$(2v(w(\pi)) =)\ 2\sqrt{w(\pi)} = \begin{cases} \gamma + \mu(1 - 3/2) & \text{if } \pi \le 1, \\ \gamma + \mu(1 - 1/2) & \text{if } \pi > 1. \end{cases}$$

Example 2 (cont.) Since IR is binding,

$$\int v(w(\pi))f(\pi|e)d\pi - g(e) = \bar{u},$$

$$\rightarrow a + b = \int_0^2 v(w(\pi))f(\pi|e_H)d\pi = \frac{\gamma}{2}.$$

Since IC is binding,

$$\int v(w(\pi))f(\pi|e_H)d\pi - g(e_H)$$

$$= \int v(w(\pi))f(\pi|e_L)d\pi - g(e_L),$$

$$\rightarrow \frac{\gamma}{2} - a = \frac{4\gamma - \mu}{8} \quad or \quad \mu = 8a.$$

Example 2 (cont.) $\gamma = 2(a+b)$ and $\mu = 8a$.

$$v(w(\pi)) = \sqrt{w(\pi)} = \begin{cases} (2\gamma - \mu)/4 = b - a & \text{if } \pi \leq 1, \\ (2\gamma + \mu)/4 = b + 3a & \text{if } \pi > 1. \end{cases}$$

$$E[w(\pi)|e_H] = (b-a)^2/2 + (b+3a)^2/2 = b^2 + 2ab + 5a^2.$$

The owner's net profit Π_H is

$$\Pi_H = E(\pi|e_H) - E[w(\pi)|e_H] = 1 - (b^2 + 2ab + 5a^2).$$

Example 2 (cont.)

The optimal wage implementing e_L is

$$w^*_{eL} = v^{-1}(\bar{u} + g(e_L)) = (b+0)^2 = b^2.$$

The expected gross profit is

$$E(\pi|e_L) = \int_0^1 \pi \frac{3}{4} d\pi + \int_1^2 \pi \frac{1}{4} d\pi = \frac{3}{4}.$$

Example 3 $\pi \in [0,3]$, $g(e_H) = a \in (0,3)$, $g(e_L) = 0$, $\bar{u} = b$, $v(w) = \sqrt{w}$, $v^{-1}(v) = v^2$. $v'(w) = 1/(2\sqrt{w})$.

For $\pi \in [0,3]$,

$$f(\pi|e_L) = \begin{cases} 1/2 & \text{if } \pi \in [0,1], \\ 1/6 & \text{if } \pi \in (1,2], \\ 1/3 & \text{if } \pi \in (2,3], \end{cases} \qquad f(\pi|e_H) = \frac{1}{3},$$

$$L(\pi) \equiv \frac{f(\pi|e_L)}{f(\pi|e_H)} = \begin{cases} 3/2 & \text{if } \pi \in [0,1], \\ 1/2 & \text{if } \pi \in (1,2], \\ 1 & \text{if } \pi \in (2,3]. \end{cases}$$

Example 3 (cont.) Let $w(\cdot)$ be an optimal solution to (10) implementing $e = e_H$. (11) is

$$(2v(w(\pi)) =)\ 2\sqrt{w(\pi)} = \begin{cases} \gamma - \mu/2 & \text{if } \pi \in [0,1], \\ \gamma + \mu/2 & \text{if } \pi \in (1,2], \\ \gamma & \text{if } \pi \in (2,3]. \end{cases}$$

Since IR is binding,

$$\int v(w(\pi)) f(\pi|e) d\pi - g(e) = \bar{u},$$

$$\rightarrow a + b = \int_0^3 v(w(\pi)) f(\pi|e_H) d\pi = \frac{\gamma}{2}.$$

Example 3 (cont.) Since IC is binding,

$$\int v(w(\pi))f(\pi|e_H)d\pi - g(e_H)$$

$$= \int v(w(\pi))f(\pi|e_L)d\pi - g(e_L),$$

$$\to \frac{\gamma}{2} - a = \frac{6\gamma - \mu}{12} \quad or \quad \mu = 12a.$$

Substituting γ and μ into $v(w(\pi))$, we have

$$v(w(\pi)) = \sqrt{w(\pi)} = \begin{cases} (2\gamma - \mu)/4 = b - 2a & \text{if } \pi \in [0,1], \\ (2\gamma + \mu)/4 = b + 4a & \text{if } \pi \in (1,2], \\ \gamma/2 = b + a & \text{if } \pi \in (2,3]. \end{cases}$$

Example 3 (cont.) The expected wage payment is

$$E[w(\pi)|e_H] = b^2 + 2ab + 7a^2.$$

The owner's net profit Π_H is

$$\Pi_H = E(\pi|e_H) - E[w(\pi)|e_H] = 3/2 - (b^2 + 2ab + 5a^2).$$

Example 3 (cont.)

The optimal wage implementing e_L is

$$w^*_{eL} = v^{-1}(\bar{u} + g(e_L)) = (b+0)^2 = b^2.$$

The expected gross profit is

$$E(\pi|e_L) = \int_0^1 \pi \frac{1}{2} d\pi + \int_1^2 \pi \frac{1}{6} d\pi + \int_2^3 \pi \frac{1}{3} d\pi = \frac{4}{3}.$$

$$\text{Eq.}(11): \quad \frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)} \right].$$

Monotone likelihood ratio property (MLRP)

$L(\pi) \equiv \dfrac{f(\pi|e_L)}{f(\pi|e_H)}$ is decreasing in π.

Fact 14.B.2 Let $w(\cdot)$ be an optimal solution to (10).

$w(\cdot)$ is increasing $\Leftrightarrow$ MLRP holds.

$$\text{Eq.}(11): \quad \frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)} \right].$$

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Fact 14.B.2 Let $w(\cdot)$ be an optimal solution to (10).

$$w(\cdot) \text{ is increasing} \quad \Leftrightarrow \quad \text{MLRP holds.}$$

Taking derivatives of both sides of (11), we have

$$-\frac{v''(w(\pi))w'(\pi)}{[v'(w(\pi))]^2} = -\mu L'(\pi) \ \rightarrow \ w'(\pi) = \frac{\mu[v'(w(\pi))]^2}{v''(w(\pi))} L'(\pi).$$

Since $\mu > 0$ and $v'' < 0$, $w'(\pi) > 0$ if and only if $L'(\pi) < 0$.

Jensen's Inequality Let x be a random variable. Let h: $\Re \to \Re$.

$$h \text{ is concave} \quad \Rightarrow \quad h(E(x)) \geq E(h(x)),$$
$$h \text{ is strictly concave} \quad \Rightarrow \quad h(E(x)) > E(h(x)).$$

Fact 14.B.3 $w(\cdot)$ is an optimal wage to implement e_H.

$E[w(\pi)|e_H] > v^{-1}(\bar{u} + g(e_H)) = w^*_{eH}$ (observable case).

Proof Since IR is binding (Lemma 1),

$$E[v(w(\pi))|e_H] = \bar{u} + g(e_H).$$

Since $v(\cdot)$ is strictly concave,

$$v(E[w(\pi)|e_H]) > E[v(w(\pi))|e_H] = \bar{u} + g(e_H),$$
$$\Rightarrow E[w(\pi)|e_H] > v^{-1}(\bar{u} + g(e_H)) = w^*_{eH}.$$

Another signal Let y be another signal of effort which is available to the owner. The density function is $f(\pi, y|e)$.

Implement e_H A condition analogous to (11)

$$\frac{1}{v'(w(\pi, y))} = \gamma + \mu \left[1 - \frac{f(\pi, y|e_L)}{f(\pi, y|e_H)} \right]. \qquad (12)$$

When y is independent of e, $f(\pi, y|e) = f_1(\pi|e) f_2(y)$. Substituting $f(\pi, y|e)$ into (12), we find that (12) is independent of y.

Intuition Suppose that the owner initially offers a wage schedule depending on y. If the owner instead offers, for each π, the certain payment $\bar{w}(\pi)$ such that

$$v(\bar{w}(\pi)) = E[v(w(\pi, y))|\pi] = \int v(w(\pi, y)) f_2(y) dy.$$

The manager's expected utility does not change. The expected wage payments becomes lower, that is, the owner is better off.

$$v(\bar{w}(\pi)) = E[v(w(\pi, y))|\pi] < v(E[w(\pi, y)|\pi]).$$

Further discussion $f(\pi, y|e) = f_1(\pi|e) f_2(y|\pi, e)$.

If f_2 does not depend on e, (12) is independent of y.

§14.C Hidden information

Setting An owner wishes to hire a manager. The random realization of **the manager's disutility** from effort is **not observable**. Now assume that effort is observable.

Hidden Information Effort $e \in [0, \infty)$, Profit: $\pi(e)$ ($\pi(0) = 0$, $\pi' > 0$, $\pi'' < 0$). Manager's utility: $u(w, e, \theta)$ ($\theta \in \Re$ is the manager's unobservable type).

Utility function $u(w, e, \theta) = v(w - g(e, \theta))$, where $g(e, \theta)$ measures the disutility of effort in monetary units.

Hidden information

Utility: $u(w, e, \theta) = v(w - g(e, \theta))$, $(v' > 0,\ v'' < 0)$.

Disutility $g(e, \theta)$**:** $g(0, \theta) = 0\ \forall \theta$.

$$g_e(e, \theta) \begin{cases} > 0 & \text{for } e > 0, \\ = 0 & \text{for } e = 0, \end{cases} \quad g_{ee}(e, \theta) > 0\ \forall e,$$

$$g_\theta(e, \theta) < 0\ \forall e,\ g_{e\theta}(e, \theta) \begin{cases} < 0 & \text{for } e > 0, \\ = 0 & \text{for } e = 0. \end{cases}$$

Higher values of θ are more productive states.
The indifference curves have the single-crossing property discussed in Ch.13.

Type: After the contract is signed, θ_H or θ_L is randomly realized $(\theta_H > \theta_L,\ \Pr(\theta_H) = \lambda \in (0, 1))$.

A contract The risk-neutral owner should insure the manager against fluctuations in his income. The contract must make the level of managerial effort responsive to the disutility incurred by the manager.

θ is observable The owner offers two wage-effort pairs (w_i, e_i) for state θ_i $(i = H, L)$.

$$\max_{w_L, e_L \geq 0, w_H, e_H \geq 0} \lambda[\pi(e_H) - w_H] + (1-\lambda)[\pi(e_L) - w_L],$$

$$s.t.\ \lambda v(w_H - g(e_H, \theta_H)) + (1-\lambda) v(w_L - g(e_L, \theta_L)) \geq \bar{u}.$$

The constraint must be binding.

F.O.C. The first-order conditions (γ is the multiplier)

$$-\lambda + \gamma\lambda v'(w_H^* - g(e_H^*, \theta_H)) = 0, \qquad (13)$$

$$-(1-\lambda) + \gamma(1-\lambda)v'(w_L^* - g(e_L^*, \theta_L)) = 0, \qquad (14)$$

$$\lambda\pi'(e_H^*) - \gamma\lambda v'(w_H^* - g(e_H^*, \theta_H))g_e(e_H^*, \theta_H) \leq 0, \qquad (15)$$

$$(1-\lambda)\pi'(e_L^*) - \gamma(1-\lambda)v'(w_L^* - g(e_L^*, \theta_L))g_e(e_L^*, \theta_L) \leq 0, \qquad (16)$$

F.O.C. The first-order conditions (γ is the multiplier)

$$-\lambda + \gamma\lambda v'(w_H^* - g(e_H^*, \theta_H)) = 0, \tag{13}$$

$$-(1-\lambda) + \gamma(1-\lambda)v'(w_L^* - g(e_L^*, \theta_L)) = 0, \tag{14}$$

$$\lambda\pi'(e_H^*) - \gamma\lambda v'(w_H^* - g(e_H^*, \theta_H))g_e(e_H^*, \theta_H) \leq 0, \tag{15}$$

$$(1-\lambda)\pi'(e_L^*) - \gamma(1-\lambda)v'(w_L^* - g(e_L^*, \theta_L))g_e(e_L^*, \theta_L) \leq 0, \tag{16}$$

The first inequality in (15) is replaced with equality if $e_H^* > 0$.

The second inequality in (16) is replaced with equality if $e_L^* > 0$.

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Insuring the agent (13) and (14) lead to

$$v'(w_H^* - g(e_H^*, \theta_H)) = v'(w_L^* - g(e_L^*, \theta_L)). \tag{17}$$

This implies that $w_H^* - g(e_H^*, \theta_H) = w_L^* - g(e_L^*, \theta_L)$.

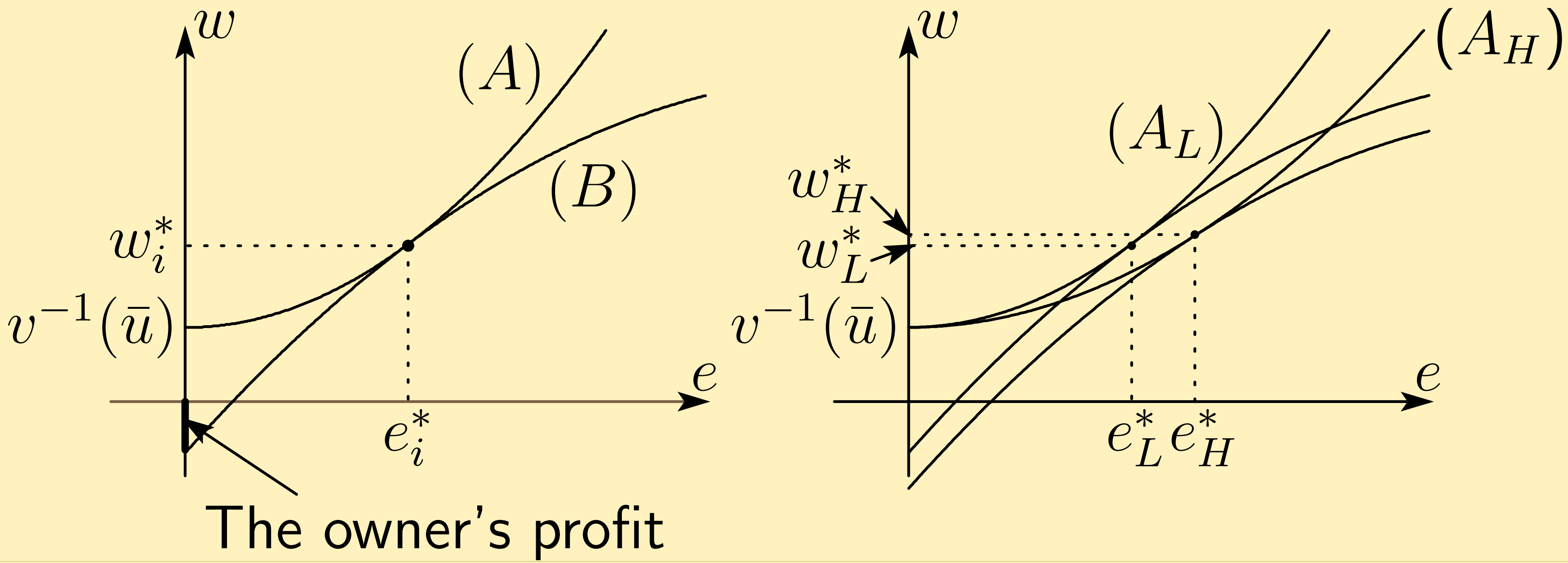
Effort level Since $g_e(0, \theta) = 0$ and $\pi'(0) > 0$, $e_i^* > 0$.

The combination of (13) and (15) ((14) and (16)) leads to

$$\pi'(e_i^*) = g_e(e_i^*, \theta_i) \text{ for } i = L, H. \tag{18}$$

Proposition 14.C.1 The optimal contract involves effort level e_i^* in state θ_i such that $\pi'(e_i^*) = g_e(e_i^*, \theta_i)$ and fully insures the manager.

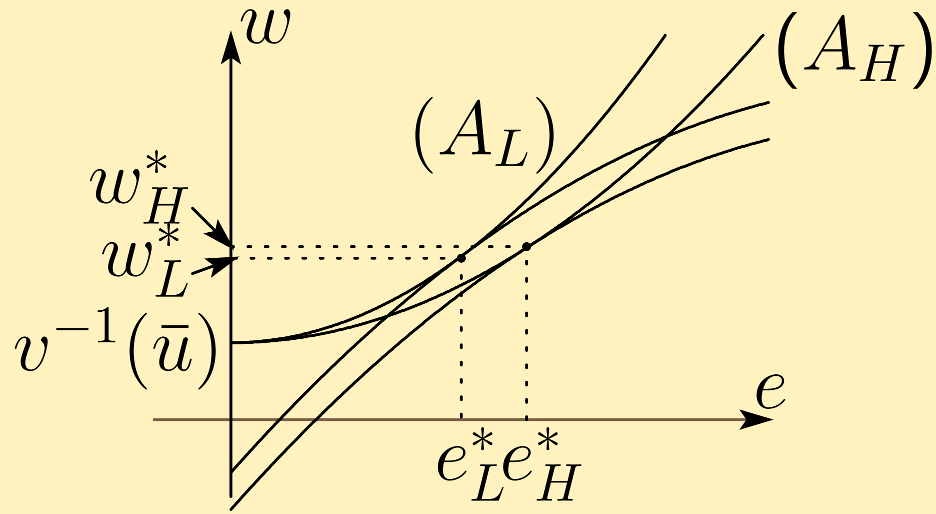
$$v'(w_H^* - g(e_H^*, \theta_H)) = v'(w_L^* - g(e_L^*, \theta_L)),$$
$$\pi'(e_i^*) = g_e(e_i^*, \theta_i) \text{ for } i = L, H,$$
$$(A) \;\; v(w - g(e, \theta_i)) = \bar{u}, \;\; (B) \;\; \pi(e) - w = \Pi_i^*.$$



Only the manager observes θ If the owner offers (w_L^*, e_L^*) and (w_H^*, e_H^*), the manager chooses (w_L^*, e_L^*) in both states, θ_L and θ_H.

In stage θ_H, the manager will lie to the owner, claiming that the state is θ_L.

What is the optimal contract?



An important result known as the **revelation principle** greatly simplifies the analysis of those types of contracting problems.

Revelation mechanism

This part is based on Laffont and Martimort (2001) *The Theory of Incentives*.

Revelation mechanism

Definition 14.C.1 Denote the set of possible states (feasible allocations) by Θ (by $\mathscr{A}$). A *direct revelation mechanism* is a mapping $h(\cdot)$ from Θ to $\mathscr{A}$ which writes as $h(\theta) = (w(\theta), e(\theta))$ for all θ belonging to Θ. The owner commits to offer the transfer $w(\tilde{\theta})$ and the effort level $e(\tilde{\theta})$ if the manager announces the value $\tilde{\theta}$ for any $\tilde{\theta}$ belonging to Θ.

Definition 14.C.2 A direct revelation mechanism $h(\cdot)$ is *truthful* if it is incentive compatible for the manager to announce his/her true type for any type.

$$v(w(\theta_L) - g(e(\theta_L), \theta_L)) \geq v(w(\theta_H) - g(e(\theta_H), \theta_L)),$$
$$v(w(\theta_H) - g(e(\theta_H), \theta_H)) \geq v(w(\theta_L) - g(e(\theta_L), \theta_H)).$$

Revelation mechanism

Definition 14.C.3 Let $\mathscr{M}$ be the message space offered to the manager by a more general mechanism. A *mechanism* is a message space $\mathscr{M}$ and a mapping $\tilde{h}(\cdot)$ from $\mathscr{M}$ to $\mathscr{A}$ which writes as $\tilde{h}(m) = (\tilde{w}(m), \tilde{e}(m))$ for all m belonging to $\mathscr{M}$.

When facing such a mechanism, the manager with type θ chooses a best message $m^*(\theta)$ that is implicitly defined as

$$v(\tilde{w}(m^*(\theta)) - g(\tilde{e}(m^*(\theta)), \theta) \geq v(\tilde{w}(\tilde{m}) - g(\tilde{e}(\tilde{m}), \theta)). \quad (19)$$

for all $\tilde{m}$ in $\mathscr{M}$.

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for all $\tilde{m}$ in $\mathscr{M}$.

The mechanism $(\mathscr{M}, \tilde{h}(\cdot))$ induces *an allocation rule* $a(\theta) = (\tilde{w}(m^*(\theta)), \tilde{e}(m^*(\theta)))$ mapping the set of types Θ into the set of allocations $\mathscr{A}$.

Revelation mechanism

Proposition 14.C.2 Any allocation rule $a(\theta)$ obtained with a mechanism $(\mathscr{M}, \tilde{h}(\cdot))$ can also be implemented with a truthful direct revelation mechanism.

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Proof The indirect mechanism $(\mathscr{M}, \tilde{h}(\cdot))$ induces an allocation rule $a(\theta) = (\tilde{w}(m^*(\theta)), \tilde{e}(m^*(\theta)))$ from Θ into $\mathscr{A}$.

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Proof The indirect mechanism $(\mathscr{M}, \tilde{h}(\cdot))$ induces an allocation rule $a(\theta) = (\tilde{w}(m^*(\theta)), \tilde{e}(m^*(\theta)))$ from Θ into $\mathscr{A}$. By composition of $\tilde{h}(\cdot)$ and $m^*(\cdot)$, we can construct a direct revelation mechanism $h(\cdot)$ mapping Θ into $\mathscr{A}$, namely $h = \tilde{h} \circ m^*$ or for all $\theta \in \Theta$

$$h(\theta) = (w(\theta), e(\theta)) \equiv \tilde{h}(m^*(\theta)) = (\tilde{w}(m^*(\theta)), \tilde{e}(m^*(\theta))).$$

Revelation mechanism

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$$h(\theta) = (w(\theta), e(\theta)) \equiv \tilde{h}(m^*(\theta)) = (\tilde{w}(m^*(\theta)), \tilde{e}(m^*(\theta))).$$

$$\Theta \xrightarrow{m^*(\cdot)} M \xrightarrow{\tilde{h}(\cdot)} \mathscr{A}$$

$$h(\cdot) = \tilde{h} \circ m^*(\cdot)$$

Revelation mechanism

Proof (cont.) We now check that the direct revelation mechanism $h(\cdot)$ is truthful. Since inequality (19),

$$v(\tilde{w}(m^*(\theta)) - g(\tilde{e}(m^*(\theta)), \theta) \geq v(\tilde{w}(\tilde{m}) - g(\tilde{e}(\tilde{m}), \theta)),$$

is true for all $\tilde{m}$, it holds in particular for $\tilde{m} = m^*(\theta')$ for all $\theta' \in \Theta$.

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$$v(\tilde{w}(m^*(\theta)) - g(\tilde{e}(m^*(\theta)), \theta) \geq v(\tilde{w}(m^*(\theta')) - g(\tilde{e}(m^*(\theta')), \theta))$$

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Proof (cont.) We now check that the direct revelation mechanism $h(\cdot)$ is truthful. Since inequality (19),

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$$v(\tilde{w}(m^*(\theta)) - g(\tilde{e}(m^*(\theta)), \theta) \geq v(\tilde{w}(m^*(\theta')) - g(\tilde{e}(m^*(\theta')), \theta))$$

Finally, using the definition of $h(\cdot)$, we have

$$v(w(\theta) - g(e(\theta), \theta)) \geq v(w(\theta') - g(e(\theta'), \theta)),$$

for all (θ, θ') in Θ. The direct revelation mechanism $h(\cdot)$ is truthful. Q.E.D.

A special case (infinite risk aversion)

Assumption (infinite risk aversion) The manager's expected utility is equal to his/her lowest utility level across the two states.
In each state, an infinitely risk-averse manager has a utility level equal to $\bar{u}$.

Owner By the revelation principle, the problem is

$$\max_{w_L, e_L \geq 0, w_H, e_H \geq 0} \lambda[\pi(e_H) - w_H] + (1-\lambda)[\pi(e_L) - w_L], \quad (20)$$

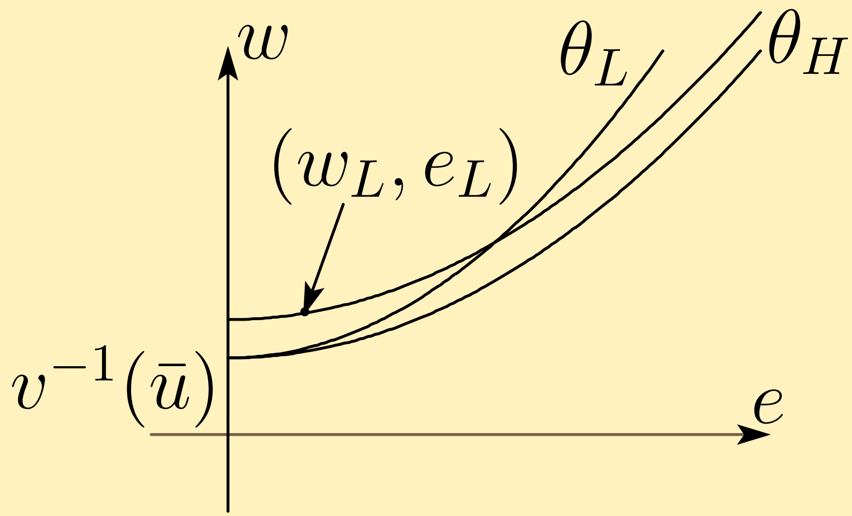
$$\begin{aligned} s.t. \quad & (i)\ w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}), \\ & (ii)\ w_H - g(e_H, \theta_H) \geq v^{-1}(\bar{u}), \\ & (iii)\ w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H), \\ & (iv)\ w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L). \end{aligned}$$

Lemma 14.C.1 We can ignore constraint (ii).

Proof By (iii), $w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H)$. By the assumption of $g(e, \theta)$ and (i),

$$w_L - g(e_L, \theta_H) \geq w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}).$$

That is, whenever (i) and (iii) are satisfied, (ii) is also satisfied.



Lemma 14.C.2 An optimal contract in problem (20) must have $w_L - g(e_L, \theta_L) = v^{-1}(\bar{u})$.

$$\max_{w_L, e_L \geq 0, w_H, e_H \geq 0} \lambda[\pi(e_H) - w_H] + (1-\lambda)[\pi(e_L) - w_L],$$
$$s.t.\ \ (i)\ w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}),$$
$$(ii)\ w_H - g(e_H, \theta_H) \geq v^{-1}(\bar{u}),$$
$$(iii)\ w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H),$$
$$(iv)\ w_L - g(e_L, \theta_L) \geq w_H - g(e_H, \theta_L).$$

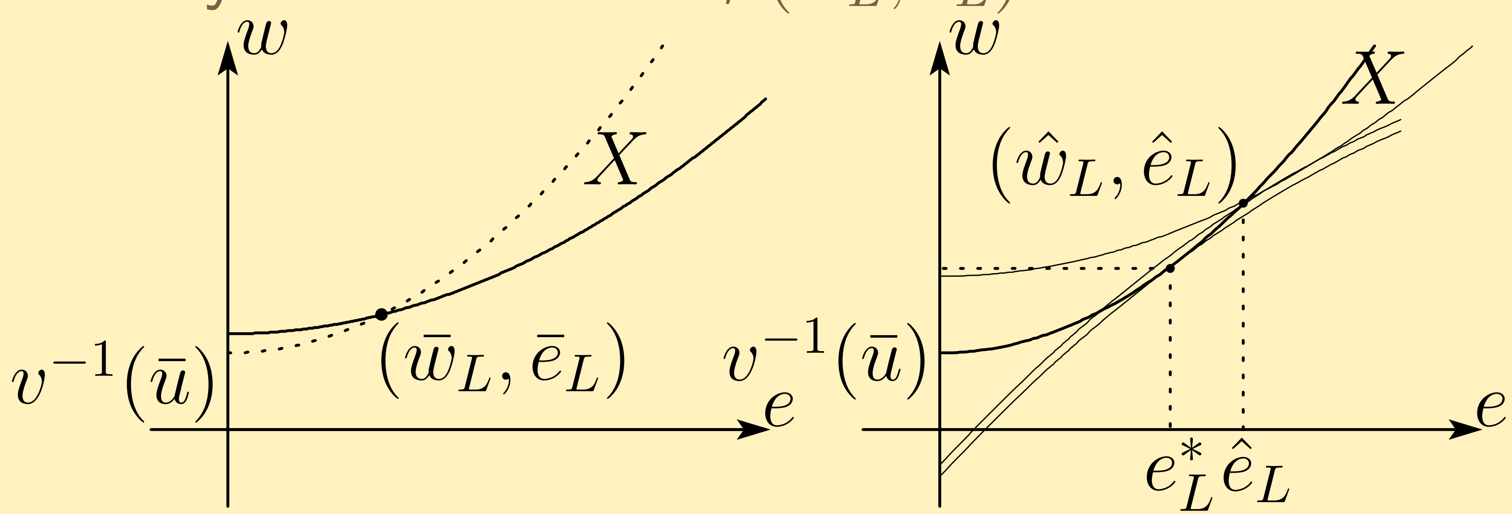
Proof Suppose that there is an optimal solution $[(w_L, e_L), (w_H, e_H)]$ in which $w_L - g(e_L, \theta_L) > v^{-1}(\bar{u})$.

A new wages $w'_L = w_L - \varepsilon$ and $w'_H = w_H - \varepsilon$, where $\varepsilon > 0$ is small enough, satisfies (i).

The new wage does not affect the incentive constraints.

Lemma 14.C.3 In any optimal contract: $e_L \leq e_L^*$ and $e_H = e_H^*$, where e_i^* would be the effort level of type θ_i if θ were observable $(i = H, L)$.

Proof By Lemma 14.C.2, (w_L, e_L) lies on the dot-line.



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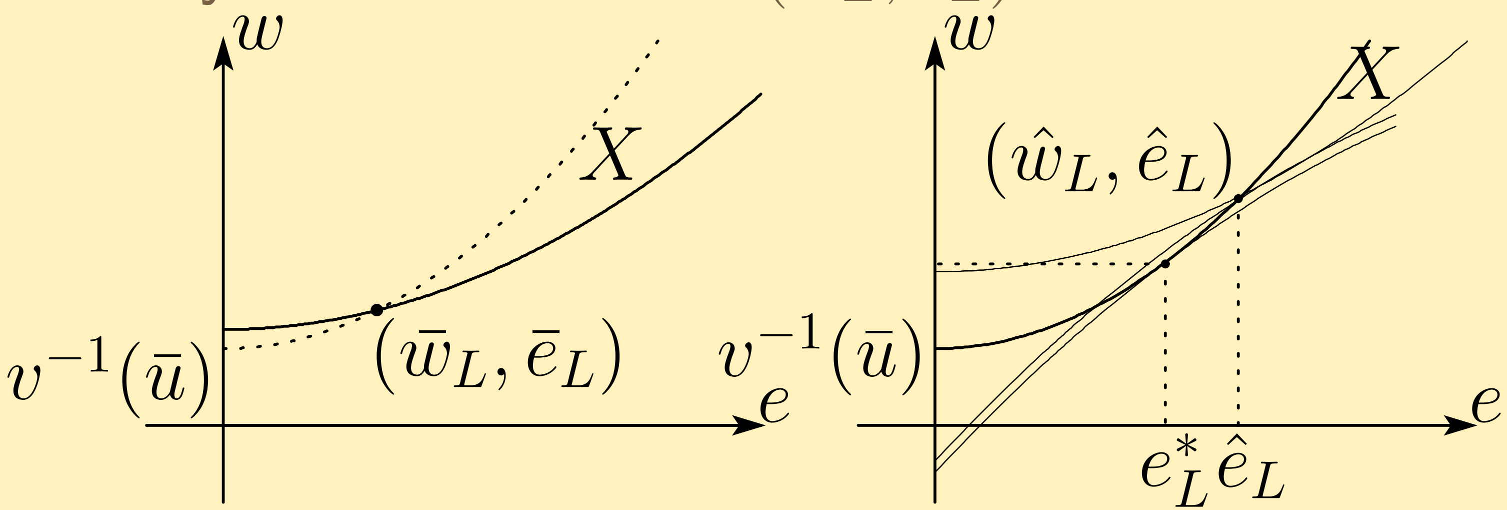
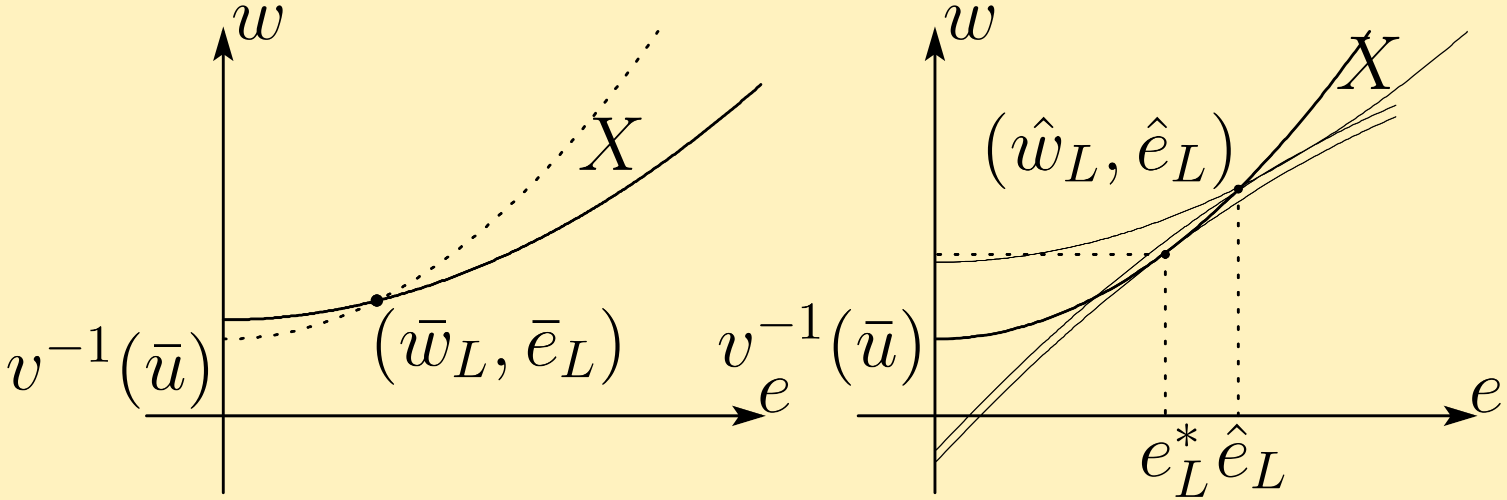


By the incentive constraints, (w_H, e_H) must lie in X.

Lemma 14.C.3 $e_L \leq e_L^*$ and $e_H = e_H^*$,

Proof By Lemma 14.C.2, (w_L, e_L) lies on the dot-line.



Suppose that $\hat{e}_L > e_L^*$. The isoprofit curve which goes through $(\hat{w}_L, \hat{e}_L)$ lies above the one which goes through (w_L^*, e_L^*). The owner can raise the profit in stage θ_L by choosing (w_L^*, e_L^*) that does not narrow X.

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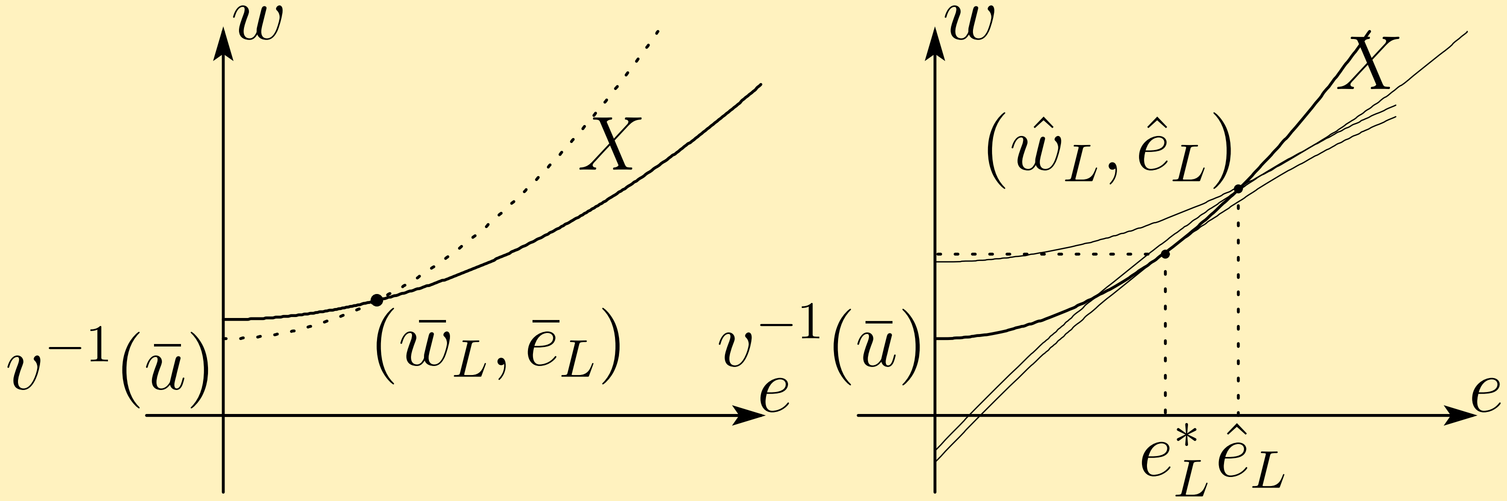
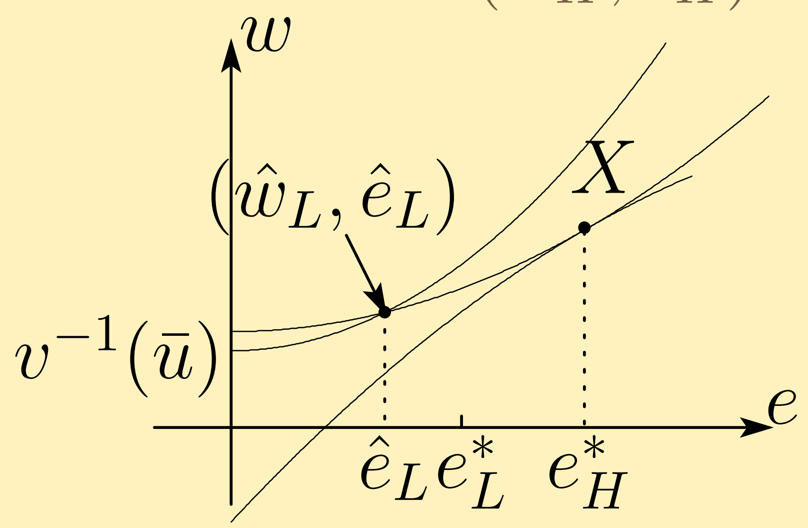
Proof By Lemma 14.C.2, (w_L, e_L) lies on the dot-line.



Suppose that $\hat{e}_L > e_L^*$. The isoprofit curve which goes through $(\hat{w}_L, \hat{e}_L)$ lies above the one which goes through (w_L^*, e_L^*). The owner can raise the profit in stage θ_L by choosing (w_L^*, e_L^*) that does not narrow X. A contract with $\hat{e}_L > e_L^*$ cannot be optimal.

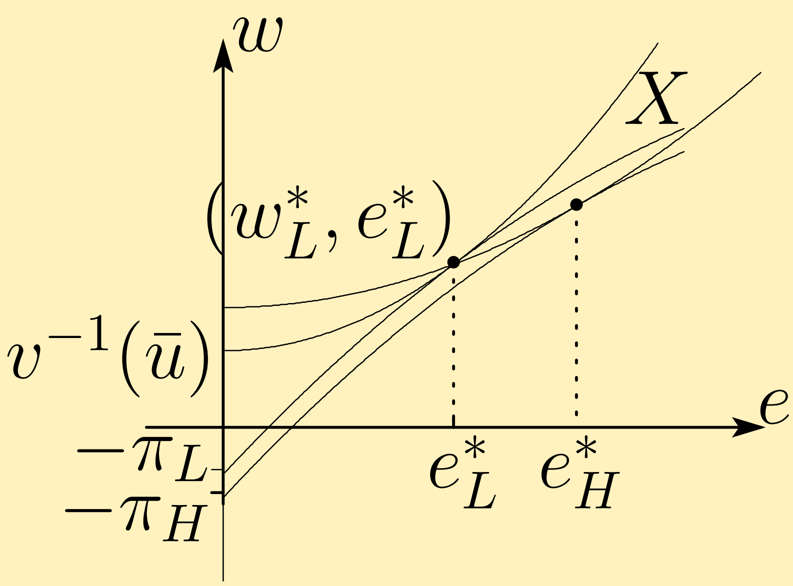
Proof (cont.) Given $(\hat{w}_L, \hat{e}_L)$ with $\hat{e}_L \leq e_L^*$ (see Figure), the owner's problem is to find (w_H, e_H) in region X.



The solution occurs at a point of tangency between the manager's state θ_H indifference curve through point $(\hat{w}_L, \hat{e}_H)$ and an isoprofit curve for the owner. This tangency occurs at $e = e_H^*$ (this is characterized by (18) in the observable case).

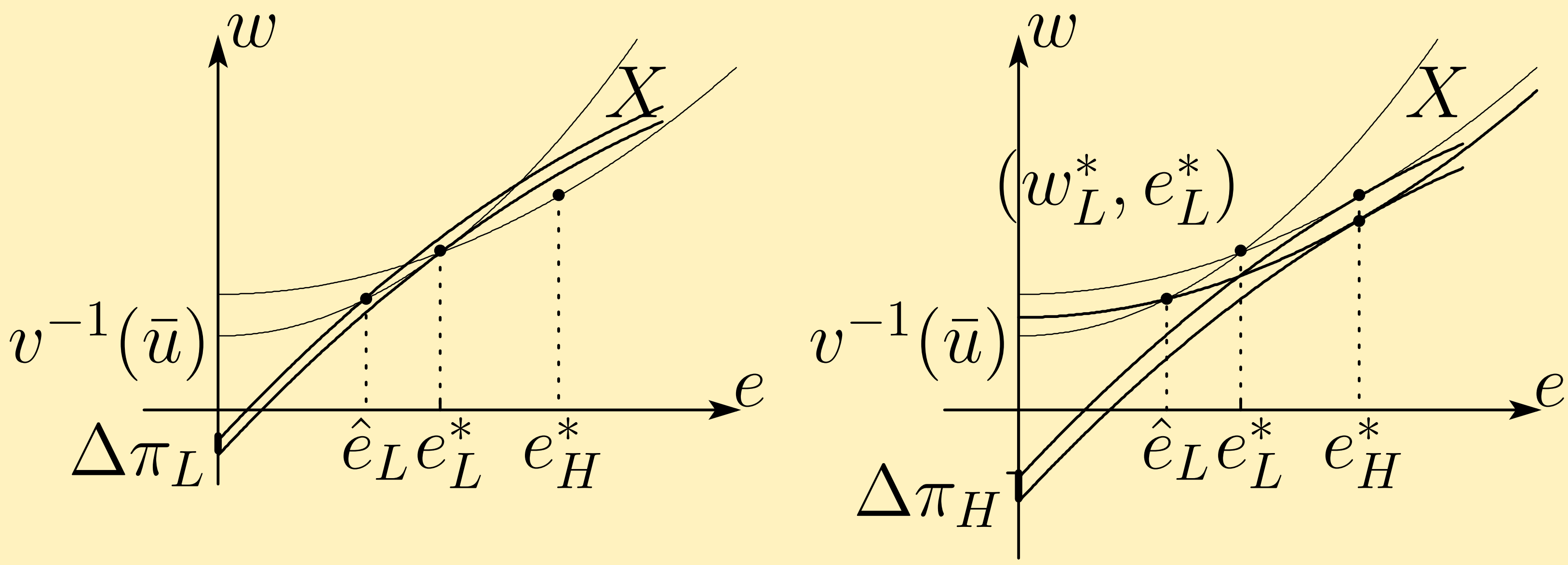
Lemma 14.C.4 In any optimal contract, $e_L < e_L^*$.

Proof (sketch) We now set $e_L = e_L^*$ (see Figure).



The expected profit is $\lambda\pi_H + (1-\lambda)\pi_L$.
Suppose that the owner slightly lowers e_L from e_L^* to $\hat{e}_L$.

Proof (cont.) Under $\hat{e}_L$, π_L decreases but π_H increases.



When the difference between e^*_L and $\hat{e}_L$ is small enough, $\Delta\pi_L$ is nearly equal to zero because the envelop theorem can be applied to this problem (e^*_L is the first-best result to maximize the owner's profit).

The optimal level of e_L The greater the likelihood of state θ_H, the more the owner is willing to distort the state θ_L outcome. The optimal level of e_L satisfies:

$$[\pi'(e_L) - g_e(e_L, \theta_L)] + \frac{\lambda}{1-\lambda}[g_e(e_L, \theta_H) - g_e(e_L, \theta_L)] = 0.$$

When $e = e_L^*$, the first term is zero and the second term is strictly negative.

Proposition 14.C.3 $e_H = e_H^*$ and $e_L < e_L^*$. The manager receives a utility greater than $\bar{u}$ in state θ_H. The owner's profit is lower than when θ is observable. The infinitely risk-averse manager's expected utility is the same as when θ is observable.