Proof of Theorem 9.1 in Tadelis (2012)

Prove it by contradiction.

Suppose that σ_i is a one-stage unimprovable but it is not optimal. Then, there exist σ'_i and information set h_i^1 such that $v_i(\sigma'_i, h_i^1) > v_i(\sigma_i, h_i^1)$. This means that there is a path starting from h_i^1 to a terminal node such that player i's payoff from the path is greater than $v_i(\sigma_i, h_i^1)$. On the path starting from h_i^1, there are n information sets of player i (the first node is h_i^1 and the last one h_i^n is a terminal node). i's payoff at the terminal node h_i^n is greater than $v_i(\sigma_i, h_i^1)$. We call those n information sets on the path (which starts from h_i^1) h_i^1, h_i^2, ..., h_i^{n-1}, and h_i^n respectively. Define player i's strategy σ_i^t $(t = 1, 2, \ldots, n-1)$ such that

$$\sigma_i^t = \begin{cases} \sigma'_i(h_i^r) & \text{if } r \leq t, \\ \sigma_i(\cdot) & \text{if } r \geq t+1, \end{cases} \tag{1}$$

where $\sigma'_i(h_i^r)$ induces the movement from the information set h_i^r to the information set h_i^{r+1} and $\sigma_i(\cdot)$ coincides with σ_i. For instance, σ_i^3 induces the path from h_i^1 to h_i^4 but makes player i play σ_i from h_i^4 to a terminal node.

σ_i is not optimal implies that (note that $\sigma_i^{n-1} = \sigma'_i$)

$$v_i(\sigma_i^{n-1}, h_i^1) > v_i(\sigma_i, h_i^1). \tag{2}$$

Note that σ_i^{n-2}, which induces the path from h_i^1 to h_i^{n-1} but makes player i play σ_i at h_i^{n-1}, fully coincides with σ_i from h_i^{n-1} to h_i^n, while σ_i^{n-1} is a one-stage deviation from σ_i at h_i^{n-1} (see the column of h_i^{n-1}).

information sets	h_i^1	h_i^2	$\cdots$	h_i^{n-2}	h_i^{n-1}	Terminal	Payoff
σ_i^{n-2}	$\sigma'_i(h_i^1)$	$\sigma'_i(h_i^2)$	$\cdots$	$\sigma'_i(h_i^{n-2})$	$\sigma_i(h_i^{n-1})$	not at h_i^n	$v_i(\sigma_i^{n-2}, h_i^{n-1})$
σ_i^{n-1}	$\sigma'_i(h_i^1)$	$\sigma'_i(h_i^2)$	$\cdots$	$\sigma'_i(h_i^{n-2})$	$\sigma'_i(h_i^{n-1})$	at h_i^n	$v_i(\sigma_i^{n-1}, h_i^{n-1})$

Because σ_i is one-stage unimprovable by assumption, it must be that

$$v_i(\sigma_i^{n-2}, h_i^{n-1}) = v_i(\sigma_i, h_i^{n-1}) \geq v_i(\sigma_i^{n-1}, h_i^{n-1}). \tag{3}$$

The last inequality comes from the definition of one-stage unimprovable (at the information set h_i^{n-1}, the inequality $v_i(\sigma_i^{n-1}, h_i^{n-1}) > v_i(\sigma_i, h_i^{n-1})$ is never satisfied for σ_i^{n-1} (definition of one-stage unimprovable), thus, $v_i(\sigma_i, h_i^{n-1}) \geq v_i(\sigma_i^{n-1}, h_i^{n-1})$).

Because σ_i^{n-2} and σ_i^{n-1} induces the same path from h_i^1 to h_i^{n-1} (see the above table (the textbook includes typo)), we have $v_i(\sigma_i^{n-2}, h_i^1) = v_i(\sigma_i^{n-2}, h_i^{n-1})$ and $v_i(\sigma_i^{n-1}, h_i^1) = v_i(\sigma_i^{n-1}, h_i^{n-1})$. Substituting them into (3), we have

$$v_i(\sigma_i^{n-2}, h_i^1) \geq v_i(\sigma_i^{n-1}, h_i^1). \tag{4}$$

From (2) and (4), we have

$$v_i(\sigma_i^{n-2}, h_i^1) > v_i(\sigma_i, h_i^1).$$

Applying the same logic to σ_i^t ($t \leq n-3$), we have the following relations ($k = 2, \ldots, n-2$)

$$v_i(\sigma_i^{n-k-1}, h_i^{n-k}) = v_i(\sigma_i, h_i^{n-k}) \geq v_i(\sigma_i^{n-k}, h_i^{n-k}),$$
$$v_i(\sigma_i^{n-k-1}, h_i^1) = v_i(\sigma_i^{n-k-1}, h_i^{n-k}),$$
$$v_i(\sigma_i^{n-k}, h_i^1) = v_i(\sigma_i^{n-k}, h_i^{n-k}).$$

information sets	h_i^1	h_i^2	$\cdots$	h_i^{n-k-1}	h_i^{n-k}	$\cdots$	$\cdots$
σ_i^{n-k-1}	$\sigma_i'(h_i^1)$	$\sigma_i'(h_i^2)$	$\cdots$	$\sigma_i'(h_i^{n-k-1})$	$\sigma_i(h_i^{n-k})$	$\sigma_i(\cdot)$	$\sigma_i(\cdot)$
σ_i^{n-k}	$\sigma_i'(h_i^1)$	$\sigma_i'(h_i^2)$	$\cdots$	$\sigma_i'(h_i^{n-k-1})$	$\sigma_i'(h_i^{n-k})$	$\sigma_i(h_i^{n-k+1})$	$\sigma_i(\cdot)$

The relations lead to

$$v_i(\sigma_i^{n-k-1}, h_i^1) \geq v_i(\sigma_i^{n-k}, h_i^1).$$

By induction, we have

$$v_i(\sigma_i^1, h_i^1) > v_i(\sigma_i, h_i^1), \tag{5}$$

however, σ_i^1 is a one-stage deviation from σ_i at the information set h_i^1 (recall the definition of σ_i^1 in (1)). This implies that (5) contradicts the assertion that σ_i is one-stage unimprovable. Therefore, if σ_i is one-stage unimprovable, it must be optimal. Q.E.D.