Proof of the Zermelo's Theorem (with several additional comments)

Let σ be the strategy profile derived through backward induction.

Let $i \in N$, and let $\hat{\sigma}_i$ be player i's strategy under which he deviates from σ_i. We now prove that $v_i(\sigma) \geq v_i(\hat{\sigma}_i, \sigma_{-i})$ by induction.

Notation For each decision node x, let $Z(x)$ be the set of terminal nodes that succeed x. Given a decision node x and $z \in Z(x)$, let $n(x, z)$ be the number of decision nodes between x and z. We also define $n(x) \equiv \max_{z \in T(x)} n(x, z)$, which is called "distance" (see the figure on page 3).

Let N be $\max\{n(x) | x \text{ is a decision node of } \Gamma\}$. Because Γ is finite, N is finite.

How to play Given $n \in \{0, 1, \ldots, N\}$, we define $\hat{\sigma}_i(\cdot; n)$ for player i's strategy such that for all player i's decision node x,

$$\hat{\sigma}_i(x; n) = \begin{cases} \sigma_i(x) & \text{if } n(x) \leq n, \\ \hat{\sigma}_i(x) & \text{if } n(x) > n. \end{cases}$$

This means as follows: Given that we set n, if the distance of decision node x is smaller than or equal to n, player i's action at x follows $\sigma_i(x)$, otherwise, his action at x follows $\hat{\sigma}_i(x)$.

The following table summarizes the strategy $\hat{\sigma}_i(\cdot; n)$ for each decision node when we set n:

Distance	0	$\cdots$	n	$n+1$	$\cdots$	$\cdots$
$\hat{\sigma}_i(\cdot; n)$	$\sigma_i(\cdot)$	$\cdots$	$\sigma_i(\cdot)$	$\hat{\sigma}_i(\cdot)$	$\cdots$	$\hat{\sigma}_i(\cdot)$

Note that $\hat{\sigma}_i(\cdot; N) = \sigma_i$ because player i's strategy is $\sigma_i(x)$ for any decision node x when $n = N$.

What we do By induction, we establish that for all $n \in \{0, \ldots, N\}$,

$$v_i(\hat{\sigma}(\cdot; n), \sigma_{-i}) \geq v_i(\hat{\sigma}_i, \sigma_{-i}),$$

which implies that $v_i(\sigma) \geq v_i(\hat{\sigma}_i, \sigma_{-i})$ because $\hat{\sigma}_i(\cdot; N) = \sigma_i$.

Step I ($n = 0$): Let y be the final decision node reached by the strategy profile $(\hat{\sigma}_i(\cdot; 0), \sigma_{-i})$.

1. If y is not i's decision node, $v_i(\hat{\sigma}_i(\cdot; 0), \sigma_{-i}) = v_i(\hat{\sigma}_i, \sigma_{-i})$ because the outcome does not differ.
2. If y is i's decision node, $v_i(\hat{\sigma}_i(\cdot; 0), \sigma_{-i}) \geq v_i(\hat{\sigma}_i, \sigma_{-i})$ due to the construction of σ through the backward induction (at y, player i must choose one of the best action(s)).

Therefore, $v_i(\hat{\sigma}_i(\cdot; 0), \sigma_{-i}) \geq v_i(\hat{\sigma}_i, \sigma_{-i})$ when $n = 0$.

Step II: Assume that $v_i(\hat{\sigma}_i(\cdot;n-1),\sigma_{-i}) \geq v_i(\hat{\sigma}_i,\sigma_{-i})$.

Let x' be the decision node on the path of strategy profile $(\hat{\sigma}_i(\cdot;n),\sigma_{-i})$ such that $n(x')=n$. Note that x' is also on the path of strategy profile $(\hat{\sigma}_i(\cdot;n-1),\sigma_{-i})$ because the actions at nodes whose distance is larger than or equal to $n+1$ in the latter strategy profile are the same with those in the former strategy profile (see the red colored $\hat{\sigma}_i(\cdot)$ whose distance is larger than or equal to $n+1$ in the following table).

Distance	0	$\cdots$	$n-1$	n	$n+1$	$\cdots$	
$\hat{\sigma}_i(\cdot;n-1)$	$\sigma_i(\cdot)$	$\cdots$	$\sigma_i(\cdot)$	$\hat{\sigma}_i(\cdot)$	$\hat{\sigma}_i(\cdot)$	$\cdots$	$\hat{\sigma}_i(\cdot)$
$\hat{\sigma}_i(\cdot;n)$	$\sigma_i(\cdot)$	$\cdots$	$\sigma_i(\cdot)$	$\sigma_i(\cdot)$	$\hat{\sigma}_i(\cdot)$	$\cdots$	$\hat{\sigma}_i(\cdot)$

Note also that for all decision nodes x'' that succeed x' (the distance of x'' is smaller than or equal to $n-1$), $\hat{\sigma}_i(x'';n)=\sigma_i(x'')=\hat{\sigma}(x'';n-1)$ (see the blue colored $\sigma_i(\cdot)$ whose distance is smaller than or equal to $n-1$ in the above table).

1. If x' is not i's decision node, $v_i(\hat{\sigma}_i(\cdot;n),\sigma_{-i}) = v_i(\hat{\sigma}_i(\cdot;n-1),\sigma_{-i})$ because the outcome does not differ.

2. If x' is i's decision node, $v_i(\hat{\sigma}_i(\cdot;n),\sigma_{-i}) \geq v_i(\hat{\sigma}_i(\cdot;n-1),\sigma_{-i})$ due to the construction of σ through the backward induction (at x', player i must choose one of the best action(s)).

Therefore, $v_i(\hat{\sigma}_i(\cdot;n),\sigma_{-i}) \geq v_i(\hat{\sigma}_i(\cdot;n-1),\sigma_{-i})$.

Also, by the induction hypothesis, $v_i(\hat{\sigma}_i(\cdot;n),\sigma_{-i}) \geq v_i(\hat{\sigma}_i(\cdot;n-1),\sigma_{-i}) \geq v_i(\hat{\sigma}_i,\sigma_{-i})$, thus, $v_i(\hat{\sigma}_i(\cdot;n),\sigma_{-i}) \geq v_i(\hat{\sigma}_i,\sigma_{-i})$. We know the fact that $\hat{\sigma}_i(\cdot;N)=\sigma_i$. Therefore, $v_i(\sigma_i,\sigma_{-i}) \geq v_i(\hat{\sigma}_i,\sigma_{-i})$, that is, the deviation does not improve the payoff of player i. Q.E.D.

See also the proof in MWG pp.272-3.