Chapter 4: Rationality and Common Knowledge

4.1 Dominance in Pure Strategies

4.1.1 Dominated Strategies

Payoff function $v_i(s)$: The payoff of player i from a profile of strategies $s = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n)$.

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The strategies (actions) chosen by the players who *are not* player i are denoted by the profile

$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$.

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To simplify the exposition, we define

$$S_{-i} \equiv S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n,$$

the set of all the strategy sets of all players who are not player i.

Section 4.1

Payoff function $v_i(s_i, s_{-i})$: The payoff of player i from a profile of strategies $s = (s_i, s_{-i})$.

Prisoner's dilemma Each of the two players has an action that is best regardless of what its opponent chooses.

Player 2

		F	M
Player 1	F	$(-4, -4)$	$(-1, -5)$
	M	$(-5, -1)$	$(-2, -2)$

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Prisoner's dilemma Playing M is worse than playing F *regardless of what its opponent does.*

		Player 2: F	Player 2: M
Player 1	F	$(-4,-4)$	$(-1,-5)$
	M	$(-5,-1)$	$(-2,-2)$

Playing M is dominated.

Definition 4.1 Let $s_i \in S_i$ and $s_i' \in S_i$ be possible strategies for player i. s_i' is **strictly dominated** by s_i if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, player i's payoff from s_i' is strictly less than that from s_i. That is,

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

We will write $s_i \succ_i s_i'$ to denote that s_i' is strictly dominated by s_i.

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Claim 4.1 A rational player will never play a strictly dominated strategy.

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1/2	L	M	H
L	6,6	2,8	0,4
M	8,2	4,4	1,3
H	4,0	3,1	2,2

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Definition 4.1 Let $s_i \in S_i$ and $s'_i \in S_i$ be possible strategies for player i. s'_i is **strictly dominated** by s_i if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, player i's payoff from s'_i is strictly less than that from s_i. That is,

$$v_i(s_i, s_{-i}) > v_i(s'_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

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4.1.2 Dominant Strategy Equilibrium

Definition 4.2 $s_i \in S_i$ is a *strictly dominant strategy* for i if every other strategy of i is strictly dominated by it, that is,

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \quad \text{for all } s_i' \in S_i,\ s_i' \neq s_i, \text{ and all } s_{-i} \in S_{-i}.$$

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Definition 4.3 The strategy profile $s^D \in S$ is a *strictly dominant strategy equilibrium* if $s_i^D \in S_i$ is a strictly dominant strategy for all $i \in N$.

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Player 1	F	$(-4, -4)$	$(-1, -5)$
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(F, F) is a dominant strategy equilibrium.

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(F, F) is a dominant strategy equilibrium.

The payoffs are $(-4, -4)$ for players 1 and 2.

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(F, F) is a dominant strategy equilibrium.

Caveat The pair of payoffs is NOT the solution.
The solution should always be described as the strategies that the players will choose (p.61).

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Proposition 4.1 If the game $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ has a strictly dominant strategy equilibrium s^D, then s^D is the unique dominant strategy equilibrium.

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Pick up a strictly dominant strategy of player i, s_i^D. By definition of s_i^D, for any $s_i' \neq s_i^D$ ($s_i' \in S_i$),

$$v_i(s_i^D, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

This also means that for any $s_i' \in S_i/\{s_i^D\}$, s_i^D is not strictly dominated by s_i', implying that s_i' is never a dominant strategy of i. Thus, player i's strictly dominant strategy is unique if it exists.

Definition 4.3 The strategy profile $s^D \in S$ is a *strictly dominant strategy equilibrium* if $s_i^D \in S_i$ is a strictly dominant strategy for all $i \in N$.

Because the game has a strictly dominant strategy equilibrium $s^D \in S$, each player has a strictly dominant strategy. By definition, each player never has more than one strictly dominant strategy, that is, the strictly dominant strategy of player i, s_i^D, is unique. Therefore, the profile of the strictly dominant strategy equilibrium $s^D = (s_1^D, \ldots, s_n^D)$ is uniquely determined.

4.1.3 Evaluating Dominant Strategy Equilibrium

Narrow applicability Example: The Battle of the Sexes

Chris

Alex		O	F
	O	$(2, 1)$	$(0, 0)$
	F	$(0, 0)$	$(1, 2)$

Narrow applicability Example: The Battle of the Sexes

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		O	F
Alex	O	$(2, 1)$	$(0, 0)$
	F	$(0, 0)$	$(1, 2)$

There is no strictly dominant strategy for each player.

If we stick to the solution concept of strict dominance, we do not have any prediction for the game.

Section 4.1

High predictability s_i^D is a strictly dominant strategy equilibrium iff

$$v_i(s_i^D, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i \text{ and all } s_{-i} \in S_{-i}.$$

The relation holds even though we add/subtract a sufficiently small value $\varepsilon > 0$ to/from the payoffs.

Pareto criterion The equilibrium payoff is not always Pareto optimal:

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The failure of Pareto optimality is NOT a failure of the solution concept.

Section 4.1

Weak dominance $s_i' \in S_i$ is *weakly dominated* by s_i if, for any possible combination of the other players' strategies, player i's payoff from s_i' is weakly less than that from s_i. That is

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i} \text{ and } s_i' \neq s_i.$$

Section 4.1

Weakly dominant s_i is a *weakly dominant strategy* for i if every other strategy of i is weakly dominated by it, that is,

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \text{ for all } s_{-i}' \in S_{-i},\ s_i' \neq s_i, \text{ and all } s_{-i} \in S_{-i}.$$

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If a weakly dominant strategy equilibrium exists, it need not be unique.

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Weakly dominant s_i is a *weakly dominant strategy* for i if every other strategy of i is weakly dominated by it, that is,

$$v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i}) \text{ for all } s'_{-i} \in S_{-i},\ s'_i \neq s_i, \text{ and all } s_{-i} \in S_{-i}.$$

Suppose that there are more than one weakly dominant strategies of player i (denote set of the strategies by S_i^D). Pick up two elements from S_i^D, s_i^{D1} and s_i^{D2}. Those elements can simultaneously satisfy the above condition if $v_i(s_i^{D1}, s_{-i}) = v_i(s_i^{D2}, s_{-i})$.

4.2 Iterated Elimination of Strictly Dominated Pure Strategies

4.2.1 Iterated Elimination and Common Knowledge of Rationality

Dominated strategy A rational player never play a dominated strategy.

Section 4.2

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Common knowledge of rationality If all the players know that each player will never play a strictly dominated strategy, they can effectively ignore those strictly dominated strategies that their opponents will never play, and their opponents can do the same thing.

Section 4.2

Common knowledge of rationality If all the players know that each player will never play a strictly dominated strategy, they can effectively ignore those strictly dominated strategies that their opponents will never play, and their opponents can do the same thing.

Iterated Elimination The players effectively eliminate their strictly dominated strategies.

Section 4.2

Iterated Elimination The players effectively eliminate their strictly dominated strategies.

Example Table 1.1.1 in Gibbons (1992, p.6)

1/2	L	M	R
U	$1, 0$	$1, 2$	$0, 1$
D	$0, 3$	$0, 1$	$2, 0$

1. M strictly dominates R.

Iterated Elimination The players effectively eliminate their strictly dominated strategies.

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1/2	L	M	R
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1. M strictly dominates R.
2. After R is eliminated, U strictly dominates D.

Iterated Elimination The players effectively eliminate their strictly dominated strategies.

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1. M strictly dominates R.
2. After R is eliminated, U strictly dominates D.
3. After D is eliminated, M strictly dominates L.

Section 4.2

Iterated Elimination The players effectively eliminate their strictly dominated strategies.

Example Table 1.1.1 in Gibbons (1992, p.6)

1/2	L	M	R
U	$1, 0$	$1, 2$	$0, 1$
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1. M strictly dominates R.
2. After R is eliminated, U strictly dominates D.
3. After D is eliminated, M strictly dominates L.

Finally, only (U, M) remains. (U, M) will be played.

Section 4.2

Iterated elimination of strictly dominated strategies

This process builds on the assumption of *common knowledge of rationality*.

Section 4.2

The process of IESDS Let S_i^k denote the strategy set of player i that survives k rounds of IESDS.
We begin the process by defining $S_i^0 = S_i$ for each i.

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1. Define $S_i^0 = S_i$ for each i, and set $k = 0$.
2. Are there players for whom there are $s_i \in S_i^k$ that are strictly dominated? If yes, go to Step 3. If not, go to Step 4.

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2. Are there players for whom there are $s_i \in S_i^k$ that are strictly dominated? If yes, go to Step 3. If not, go to Step 4.
3. For all the players $i \in N$, remove any strategies $s_i \in S_i^k$ that are strictly dominated. Set $k = k + 1$, and define a new game with strategy sets S_i^k that *do not include* the strictly dominated strategies that have been removed. Go back to Step 2.

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2. Are there players for whom there are $s_i \in S_i^k$ that are strictly dominated? If yes, go to Step 3. If not, go to Step 4.
3. For all the players $i \in N$, remove any strategies $s_i \in S_i^k$ that are strictly dominated. Go back to Step 2.
4. The remaining strategies in S^k are reasonable predictions for behavior.

4.2.2 Example: Cournot Duopoly

Cournot duopoly Firms 1 and 2 produce homogenous goods. The cost function of firm i is $c_i(q_i) = cq_i$ ($i \in \{1, 2\}$), where $c(\in (0, a))$ is a positive constant and q_i is its quantity. Each firm simultaneously sets its quantity. The price in the market, p, is given by

$$p = \begin{cases} a - (q_1 + q_2) & \text{if } q_1 + q_2 < a, \\ 0 & \text{if } q_1 + q_2 \geq a. \end{cases}$$

The objective of firm i is to maximize its own profit:

$$\pi_i(q_1, q_2) = \begin{cases} (a - (q_1 + q_2) - c)q_i & \text{if } q_1 + q_2 < a, \\ -cq_i & \text{if } q_1 + q_2 \geq a. \end{cases}$$

Cournot duopoly The profit of firm i is

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The partial derivative of firm i's profit function with respect to q_i is

$$\frac{\partial \pi_i(q_1, q_2)}{\partial q_i} = \begin{cases} a - 2q_i - q_j - c & \text{if } q_1 + q_2 < a, \\ -c & \text{if } q_1 + q_2 \geq a. \end{cases}$$

For q_j, the optimal quantity of firm i is

$$q_i(q_j) = \begin{cases} (a - c - q_j)/2 & \text{if } q_j < a - c, \\ 0 & \text{if } q_j \geq a - c. \end{cases}$$

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For q_j, the optimal quantity of firm i is

$$q_i(q_j) = \begin{cases} \dfrac{a - c - q_j}{2} & \text{if } q_j < a - c, \\ 0 & \text{if } q_j \geq a - c. \end{cases}$$

Because $q_j \geq 0$, $q_i(q_j) \leq (a - c)/2$ for any q_j.

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Because $q_j \geq 0$, $q_i(q_j) \leq (a - c)/2$ for any q_j. That is, producing $q_i(> (a - c)/2)$ is excessive for any $q_j \geq 0$, which implies that $q_i(> (a - c)/2)$ is strictly dominated by $q_1 = (a - c)/2$.

Cournot duopoly The profit of firm i is

$$\pi_i(q_1, q_2) = \begin{cases} (a-(q_1+q_2)-c)q_i & \text{if } q_1+q_2 < a, \\ -cq_i & \text{if } q_1+q_2 \geq a. \end{cases}$$

For q_j, the optimal quantity of firm i is

$$q_i(q_j) = \begin{cases} \dfrac{a-c-q_j}{2} & \text{if } q_j < a-c, \\ 0 & \text{if } q_j \geq a-c. \end{cases}$$

$q_i(> (a-c)/2)$ is strictly dominated by $q_1 = (a-c)/2$. The strategy set of firm i changes to $S_i = [0, (a-c)/2]$ by IESDS.

Section 4.2

Cournot duopoly The profit of firm i is

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For q_j, the optimal quantity of firm i is

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Under the strategy set $S_i = [0, (a-c)/2]$, $(a-c)/4 \leq q_i(q_j) \leq (a-c)/2$. Producing $q_i(< (a-c)/4)$ is insufficient for any $q_j \in [0, (a-c)/2]$.

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Under the strategy set $S_i = [0, (a-c)/2]$, producing $q_i (< (a-c)/4)$ is strictly dominated by $q_i = (a-c)/4$. The strategy set of firm i changes to $S_i = [(a-c)/4, (a-c)/2]$ by IESDS.

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The strategy set of firm i changes to $S_i = [(a-c)/4, (a-c)/2]$ by IESDS.
Applying IESDS repeatedly, we find the the strategy set of firm i converges to $S_i = 30$.

4.2.3 Evaluating IESDS

Existence of an IESDS solution An IESDS solution always exists although the predictability of what will happen depends on games.

Section 4.2

Existence of an IESDS solution An IESDS solution always exists although the predictability of what will happen depends on games. IESDS implied the survival of a unique strategy in the Cournot duopoly discussed above.

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The Battle of the Sexes

		Chris	
		O	F
Alex	O	$(2,1)$	$(0,0)$
	F	$(0,0)$	$(1,2)$

Existence of an IESDS solution An IESDS solution always exists although the predictability of what will happen depends on games.
The Battle of the Sexes

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Alex	O	$(2, 1)$	$(0, 0)$
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IESDS does not eliminate any strategy, leading to the conclusion "anything can happen."

Section 4.2

IESDS and strict dominance If for a game $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ s^* is a strict dominant strategy equilibrium, then s^* uniquely survives IESDS.

IESDS and strict dominance If for a game $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ s^* is a strict dominant strategy equilibrium, then s^* uniquely survives IESDS.
If $s^* = (s_1^*, \ldots, s_n^*)$ is a strict dominant strategy equilibrium then, by definition, for every player i all other strategies s_i' are strictly dominated by s_i^*.

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If $s^* = (s_1^*, \ldots, s_n^*)$ is a strict dominant strategy equilibrium then, by definition, for every player i all other strategies s_i' are strictly dominated by s_i^*. This implies that after one stage of elimination we will be left with a single profile of strategies, which is exactly s^*.

4.3 Beliefs, Best Responses, and Rationalizability

4.3.1 The Best Response

The best response The Battle of the Sexes

		Chris	
		O	F
Alex	O	$(2,1)$	$(0,0)$
	F	$(0,0)$	$(1,2)$

The best response The Battle of the Sexes

		Chris O	Chris F
Alex	O	$(2,1)$	$(0,0)$
Alex	F	$(0,0)$	$(1,2)$

The best choice of Alex depends on what Chris will do.

Alex would rather go to the opera (the football) if Chris goes to the opera (the football).

The best response The Battle of the Sexes

		Chris	
		O	F
Alex	O	$(2,1)$	$(0,0)$
	F	$(0,0)$	$(1,2)$

Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \;\; \forall\, s_i' \in S_i.$$

The best response The Battle of the Sexes

		Chris O	Chris F
Alex	O	$(2,1)$	$(0,0)$
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Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i}) \;\; \forall \; s'_i \in S_i.$$

Claim 4.2 A rational player who believes that his opponents are playing some $s_{-i} \in S_{-i}$ will always choose a best response to s_{-i}.

The best response The Battle of the Sexes

		Chris O	Chris F
Alex	O	$(2,1)$	$(0,0)$
Alex	F	$(0,0)$	$(1,2)$

Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \;\; \forall \; s_i' \in S_i.$$

In the Battle of the Sexes, $s_A = O$ is the best response of Alex to $s_C = O$. $s_A = F$ is the best response of Alex to $s_C = F$.

Section 4.3

Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i}) \;\; \forall \; s'_i \in S_i.$$

Proposition 4.3 If s_i is a strictly dominated strategy for player i, then it cannot be a best response to any $s_{-i} \in S_{-i}$.

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Proposition 4.3 If s_i is a strictly dominated strategy for player i, then it cannot be a best response to any $s_{-i} \in S_{-i}$.

If s_i is strictly dominated, there exists some $s_i' \succ_i s_i$ such that $v_i(s_i', s_{-i}) > v_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$. This implies that there is no $s_{-i} \in S_{-i}$ for which $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i})$, and thus that s_i cannot be a best response to any $s_{-i} \in S_{-i}$.

Section 4.3

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Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

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If s^* is a strictly dominant strategy equilibrium, it uniquely survives IESDS (Proposition 4.2). So, it is sufficient to prove the proposition for strategies that uniquely survive IESDS.

Section 4.3

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$$v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i}) \;\; \forall \; s'_i \in S_i.$$

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and choose some $i \in N$.

Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s'_i, s_{-i}) \;\; \forall \; s'_i \in S_i.$$

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and choose some $i \in N$. Suppose in negation to the proposition that s^* is NOT a best response to s^*_{-i}.

Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \;\; \forall \; s_i' \in S_i.$$

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and choose some $i \in N$. Suppose in negation to the proposition that s^* is NOT a best response to s^*_{-i}. In other words, we prove the proposition by contradiction.

Section 4.3

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$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \;\; \forall \; s_i' \in S_i.$$

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and choose some $i \in N$. Suppose in negation to the proposition that s^* is NOT a best response to s^*_{-i}. This supposition implies that there exists an $s_i' \in S_i \backslash \{s_i^*\}$ such that $v_i(s_i', s^*_{-i}) > v_i(s_i^*, s^*_{-i})$.

Definition 4.5 The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \;\; \forall \; s_i' \in S_i.$$

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and also that there exists an $s_i' \in S_i \backslash \{s_i^*\}$ such that $v_i(s_i', s^*_{-i}) > v_i(s_i^*, s^*_{-i})$.

Section 4.3

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Suppose that s^* uniquely survives IESDS, and also that there exists an $s_i' \in S_i \backslash \{s_i^*\}$ such that $v_i(s_i', s^*_{-i}) > v_i(s_i^*, s^*_{-i})$. Let $S_i' \subset S_i$ be the set of all such s_i' for which $v_i(s_i', s^*_{-i}) > v_i(s_i^*, s^*_{-i})$.

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and also that there exists an $s'_i \in S_i \backslash \{s^*_i\}$ such that $v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$. Let $S'_i \subset S_i$ be the set of all such s'_i for which $v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$. $(*)$ Because s'_i was eliminated while s^*_{-i} was not, there must be some s''_i such that $v_i(s''_i, s^*_{-i}) > v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$, implying that $s''_i \in S'_i$.

Section 4.3

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Suppose that s^* uniquely survives IESDS, and also that there exists an $s'_i \in S_i \backslash \{s^*_i\}$ such that $v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$. Let $S'_i \subset S_i$ be the set of all such s'_i for which $v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$. $(*)$ Because s'_i was eliminated while s^*_{-i} was not, there must be some s''_i such that $v_i(s''_i, s^*_{-i}) > v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$, implying that $s''_i \in S'_i$. Because the game is finite, an induction argument on S'_i implies that there exists a strategy $s'''_i \in S'_i$ that must survive IESDS (we cannot repeat a kind of the procedure $(*)$ infinitely).

Proposition 4.4 If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s^* is a best response to s^*_{-i} $\forall i \in N$.

Suppose that s^* uniquely survives IESDS, and also that there exists an $s'_i \in S_i \backslash \{s^*_i\}$ such that $v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$. Let $S'_i \subset S_i$ be the set of all such s'_i for which $v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$. $(*)$ Because s'_i was eliminated while s^*_{-i} was not, there must be some s''_i such that $v_i(s''_i, s^*_{-i}) > v_i(s'_i, s^*_{-i}) > v_i(s^*_i, s^*_{-i})$, implying that $s''_i \in S'_i$. There exists a strategy $s'''_i \in S'_i$ that must survive IESDS. This contradicts to the fact that s^*_i uniquely survives IESDS.

4.3.2 Beliefs and Best-Response Correspondences

Definition 4.6 A *belief* of player i is a possible profile of his opponents' strategies, $s_{-i} \in S_{-i}$.

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(ex.) Cournot duopoly: The profit of firm i is

$$\pi_i(q_1, q_2) = \begin{cases} (a - (q_1 + q_2) - c)q_i & \text{if } q_1 + q_2 < a, \\ -cq_i & \text{if } q_1 + q_2 \geq a. \end{cases}$$

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For any q_j, the optimal quantity of firm i is

$$q_i(q_j) = \begin{cases} (a - c - q_j)/2 & \text{if } q_j < a - c, \\ 0 & \text{if } q_j \geq a - c. \end{cases}$$

This is the best-response function of firm i.
The list of best responses maps beliefs into a choice of action.

Section 4.3

Definition 4.6 A *belief* of player i is a possible profile of his opponents' strategies, $s_{-i} \in S_{-i}$.

1/2	L	C	R
U	$3, 3$	$5, 1$	$6, 2$
M	$4, 1$	$8, 4$	$3, 6$
D	$4, 0$	$9, 6$	$6, 8$

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U	$3, 3$	$5, 1$	$6, 2$
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Player 1 has *more than one* best-response strategy.

Best-response correspondence Player 1 has *more than one* best-response strategy.

1/2	L	C	R
U	$3, 3$	$5, 1$	$6, 2$
M	$4, 1$	$8, 4$	$3, 6$
D	$4, 0$	$9, 6$	$6, 8$

Definition 4.7 The *best-response correspondence* of player i selects for each $s_{-i} \in S_{-i}$ a subset $BR_i(s_{-i}) \subset S_i$ where each strategy $s_i \in BR_i(s_{-i})$ is a best response to s_{-i}.

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(ex.) 'Strange' Cournot duopoly ($c = 0$): The profit of firm i is

$$\pi_i(q_1, q_2) = \begin{cases} (a - (q_1 + q_2))q_i & \text{if } q_1 + q_2 < a, \\ 0 & \text{if } q_1 + q_2 \geq a. \end{cases}$$

Definition 4.7 The *best-response correspondence* of player i selects for each $s_{-i} \in S_{-i}$ a subset $BR_i(s_{-i}) \subset S_i$ where each strategy $s_i \in BR_i(s_{-i})$ is a best response to s_{-i}.

(ex.) 'Strange' Cournot duopoly ($c = 0$): The profit of firm i is

$$\pi_i(q_1, q_2) = \begin{cases} (a - (q_1 + q_2))q_i & \text{if } q_1 + q_2 < a, \\ 0 & \text{if } q_1 + q_2 \geq a. \end{cases}$$

For any q_j, the optimal quantity of firm i is

$$q_i(q_j) = \begin{cases} (a - q_j)/2 & \text{if } q_j < a, \\ \text{any nonnegative } q_i & \text{if } q_j \geq a. \end{cases}$$

Remember the definition of best response.