Chapter 5: Pinning Down Beliefs: Nash Equilibrim

Outline

- Nash equilibrium (Section 5.1)
- 5 (modified) examples (Section 5.2)
- Brief discussion on Nash equilibrium (MWG Chap.8)

5.1 Nash Equilibrium in Pure Strategies

Definition 5.1 The pure-strategy profile $s^* = (s_1^*, \dots, s_n*) \in S$ is a **Nash equilibrium** if s_i^* is a best response to s_{-i}^*, for all $i \in N$, that is,

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i', s_{-i}^*) \quad \text{for all } s_i' \in S_i \text{ and all } i \in N.$$

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Remember the definition of best response: The strategy $s_i \in S_i$ is player i's *best response* to his opponents' strategies $s_{-i} \in S_{-i}$ if

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \;\; \forall \; s_i' \in S_i.$$

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Question Find Nash *equilibria*.

1/2	L	M	H
L	6,6	2,8	0,4
M	8,2	4,4	1,3
H	4,0	3,1	2,2

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Proposition 5.1 Consider a strategy profile $s^* = (s_1^*, \ldots, s_n*) \in S$. If s^* is either

1. a strict dominant strategy equilibrium, or
2. the unique survivor of IESDS,

then s^* is the unique Nash equilibrium.

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Remark The requirements for a Nash equilibrium are

1. Each player is playing *a best response to his beliefs*.
2. The beliefs of the players about their opponents are *correct*.

Example (1)

Cournot Duopoly Firms 1 and 2 simultaneously set their quantities, q_1 and q_2, respectively.

- $P(Q) = a - Q$: The market price where $Q = q_1 + q_2$.
- $C_i(q_i) = c_i q_i$: The total cost to firm i, $c_i \in (0, a/2)$.

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Nash equilibrium In the context of this model, the equilibrium quantity of firm i is written as

$$q_i^* = \arg \max_{0 \le q_i < \infty} \pi_i(q_i, q_j^*) = \arg \max_{0 \le q_i < \infty} (a - (q_i + q_j^*) - c_i) q_i.$$

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Solving the problem, we have

$$q_i = \frac{a - c_i - q_j^*}{2} \quad \rightarrow \quad q_i^* = \frac{a - 2c_i + c_j}{3}.$$

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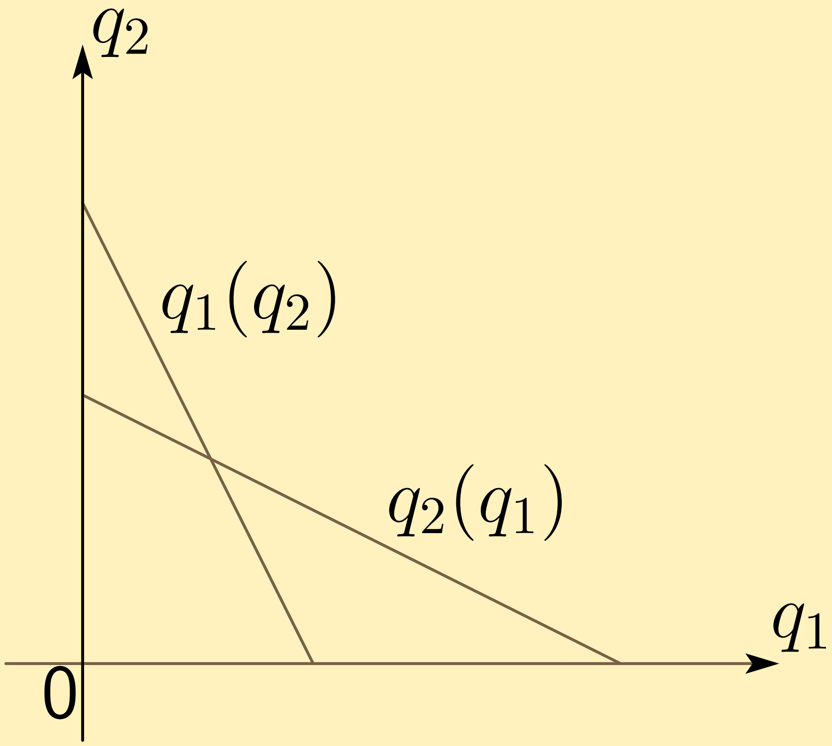

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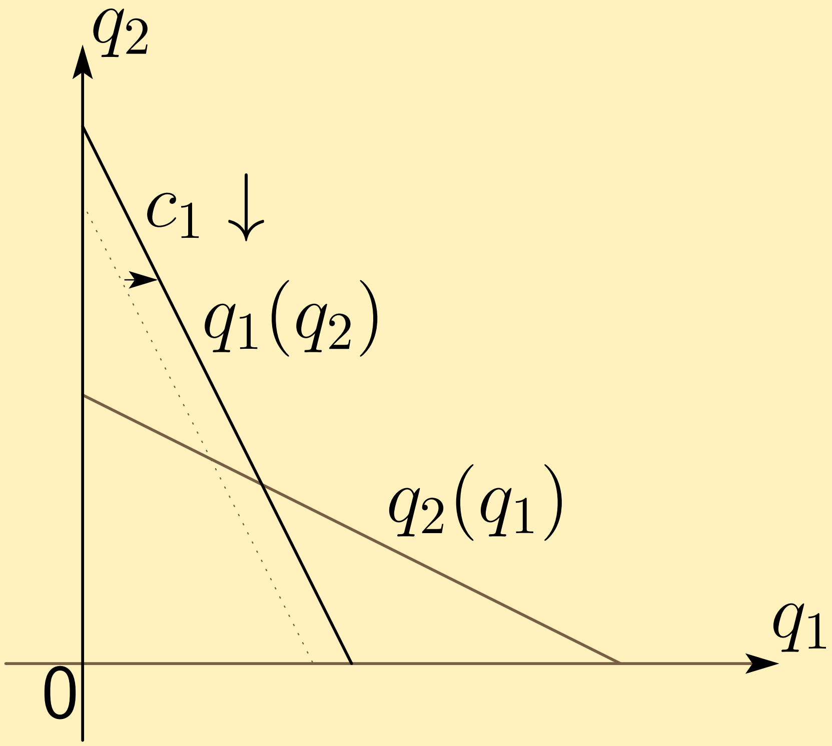

Example (2)

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Demand The market demand is denoted by $Q_i(p_i, p_j)$ $(i = 1, 2,\ j \neq i)$.

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Profits The profit of firm i is given by $(i = 1, 2)$

$$\Pi^i(p_i, p_j) = (p_i - c)Q_i(p_i, p_j),$$

where firm i faces demand

$$Q_i(p_i, p_j) = \begin{cases} Q(p_i) & \text{if } \ p_i < p_j, \\ Q(p_i)/2 & \text{if } \ p_i = p_j, \\ 0 & \text{if } \ p_i > p_j, \end{cases}$$

where $Q'(p) < 0$ and $Q''(p) < 0$ to secure the S.O.C. in the case of monopoly.

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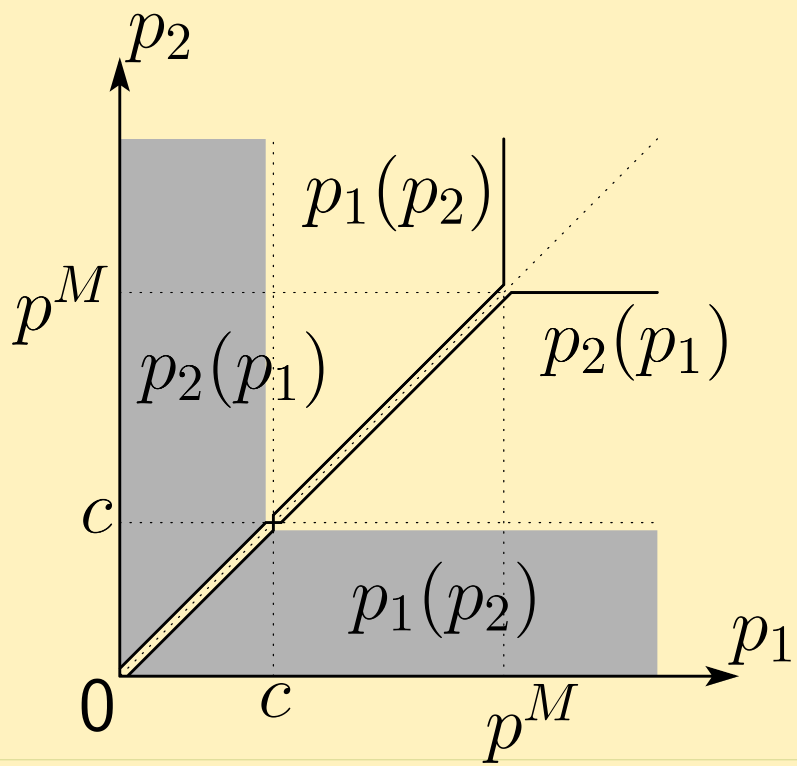

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Social optimum The objective to achieve the social optimum and the first-order condition are

$$\max_{G} \; Gv(G) - cG, \quad v(G) + Gv'(G) - c = 0.$$

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Voters are uniformly distributed along the interval, $[0, 1]$.

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Candidates The number of candidates is 3.
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Timing The candidates simultaneously choose their locations. What happens if the number of candidates is 3?

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Equilibrium Continuously many equilibria exist!

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Equilibrium If $n = 3$, no pure strategy equilibrium exists!, otherwise, a pure strategy equilibrium exists.

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