Chapter 6: Mixed Strategies

Outline

- Strategies, Beliefs, and Expected Payoffs (Section 6.1)
- Mixed-Strategy Nash Equilibrium (Section 6.2)
- 4 examples
- Nash's existence theorem (Section 6.4)
  Please see the supplement if you are interested in the proof.

6.1 Nash Equilibrium in Pure Strategies

6.1.1 Finite Strategy Sets

Motivating example (Matching pennies)

$1/2$	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

6.1 Nash Equilibrium in Pure Strategies
6.1.1 Finite Strategy Sets

Motivating example (Matching pennies)

1/2	H	T
H	$\underline{\mathbf{1}}, -1$	$-1, \underline{\mathbf{1}}$
T	$-1, \underline{\mathbf{1}}$	$\underline{\mathbf{1}}, -1$

No pure strategy Nash equilibrium exists.

Section 6.1

Definition 6.1 Let $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im}\}$ be player i's finite set of pure strategies. Define ΔS_i as the **simplex** of S_i, which is the set of all probability distribution over S_i. A **mixed strategy** for player i is an element $\sigma_i \in \Delta S_i$, so that $\sigma_i = \{\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{im})\}$ is a probability distribution over S_i, where $\sigma_i(s_{ik})$ is the probability that player i plays s_{ik} $(k = 1, 2, \ldots, m)$.

1. $\sigma_i(s_i) \geq 0$ for all $s_i \in S_i$; 2. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

Definition 6.1 A **mixed strategy** for player i is an element $\sigma_i \in \Delta S_i$, so that $\sigma_i = \{\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{im})\}$ is a probability distribution over S_i, where $\sigma_i(s_{ik})$ is the probability that player i plays s_{ik} $(k = 1, 2, \ldots, m)$.

1/2	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

(ex.) $\sigma_1 = (\sigma_1(H), \sigma_1(L)) = (1/3, 2/3)$.

Player 1 plays H with prob. 1/3 and L with prob. 2/3.

Section 6.1

Definition 6.1 A **mixed strategy** for player i is an element $\sigma_i \in \Delta S_i$, so that $\sigma_i = \{\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{im})\}$ is a probability distribution over S_i, where $\sigma_i(s_{ik})$ is the probability that player i plays s_{ik} $(k = 1, 2, \ldots, m)$.

Definition 6.2 Given a mixed strategy $\sigma_i(\cdot)$ for player i, we will say that a pure strategy $s_i \in S_i$ is in **the support of** σ_i iff it occurs with positive probability, that is, $\sigma_i(s_i) > 0$.

6.1.2 Continuous Strategy Sets

Definition 6.3 Let S_i be player i's pure strategy set and assume that S_i is an interval. A **mixed strategy** for player i is a cumulative distribution function $F_i : S_i \to [0, 1]$, where $F_i(x) = \Pr\{s_i \leq x\}$. If $F_i(\cdot)$ is differentiable with density $f_i(\cdot)$, then we say that $s_i \in S_i$ is in the support of $F_i(\cdot)$ if $f_i(s_i) > 0$.

Definition 6.3 A **mixed strategy** for player i is a cumulative distribution function $F_i : S_i \to [0, 1]$, where $F_i(x) = \Pr\{s_i \le x\}$.

(ex) The Cournot duopoly with a capacity constraint of 100 unit of production, so that $S_i = [0, 100]$ ($i = 1, 2$).

$$F_i(s_i) = \begin{cases} 0 & \text{for } s_i \in [0, 30) \\ \dfrac{s_i - 30}{20} & \text{for } s_i \in [30, 50] \\ 1 & \text{for } s_i \in (50, 100] \end{cases}$$

Definition 6.3 A **mixed strategy** for player i is a cumulative distribution function $F_i : S_i \to [0, 1]$, where $F_i(x) = \Pr\{s_i \leq x\}$.

(ex) The Cournot duopoly with a capacity constraint of 100 unit of production, so that $S_i = [0, 100]$ $(i = 1, 2)$.

$$f_i(s_i) = \begin{cases} 0 & \text{for } s_i \in [0, 30) \\ \dfrac{1}{20} & \text{for } s_i \in [30, 50] \\ 1 & \text{for } s_i \in (50, 100] \end{cases}$$

Player i chooses a quantity between 30 to 50 using a uniform distribution.

6.1.3 Beliefs and Mixed Strategies

Definition 6.4 A **belief** for player i is given by a *probability distribution* $\pi_i \in \Delta S_{-i}$ over the strategies of his opponents. We denote by $\pi_i(s_{-i})$ the probability player i assigns to his opponents playing $s_{-i} \in S_{-i}$.

In the matching pennies game, the belief of player 1 is represented by $(\pi_1(H_2), \pi_1(T_2))$, where $\pi_1(H_2), \pi_1(T_2) \geq 0$ and $\pi_1(H_2) + \pi_1(T_2) = 1$ (each subscript represents player i).

6.1.4 Expected Payoffs

Definition 6.5 The **expected payoff** of player i when he chooses the pure strategy $s_i \in S_i$ and his opponents play the mixed strategy $\sigma_{-i} \in \Delta S_{-i}$ is

$$v_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) v_i(s_i, s_{-i}).$$

Definition 6.5 Similarly, the expected payoff of player i when he chooses the mixed strategy $\sigma_i \in \Delta S_i$ and his opponents play the mixed strategy $\sigma_{-i} \in \Delta S_{-i}$ is

$$
\begin{aligned}
v_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i) v_i(s_i, \sigma_{-i}) \\
&= \sum_{s_i \in S_i} \left(\sum_{s_{-i} \in S_{-i}} \sigma_i(s_i) \sigma_{-i}(s_{-i}) v_i(s_i, s_{-i}) \right).
\end{aligned}
$$

Example The rock-paper-scissors game

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Example The rock-paper-scissors game

$1/2$	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Suppose that player 2's mixed strategy is

$$\sigma_2 = (\sigma_2(R), \sigma_2(P), \sigma_2(S)) = (1/3, 2/3, 0).$$

Example The rock-paper-scissors game

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Suppose that player 2's mixed strategy is
$\sigma_2 = (\sigma_2(R), \sigma_2(P), \sigma_2(S)) = (1/3, 2/3, 0)$.

The expected payoffs of player 1 from his strategies are

$$\begin{aligned} v_1(R, \sigma_2) &= (1/3) \times 0 + (2/3) \times (-1) + 0 \times 1 = -2/3, \\ v_1(P, \sigma_2) &= (1/3) \times 1 + (2/3) \times 0 + 0 \times (-1) = 1/3, \\ v_1(S, \sigma_2) &= (1/3) \times (-1) + (2/3) \times 1 + 0 \times 0 = 1/3. \end{aligned}$$

6.2 Mixed-Strategy Nash Equilibrium

Definition 6.6 The mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if σ_i^* is a best response to σ_{-i}^*, for all $i \in N$. That is, for all $i \in N$,

$$v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta S_i.$$

Definition 5.1 The pure-strategy profile $s^* = (s_1^*, \ldots, s_n*) \in S$ is a Nash equilibrium if s_i^* is a best response to s_{-i}^*, for all $i \in N$, that is,

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i', s_{-i}^*) \quad \text{for all } s_i' \in S_i \text{ and all } i \in N.$$

Section 6.2

Definition 6.6 The mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a *Nash equilibrium* if σ_i^* is a best response to σ_{-i}^*, for all $i \in N$. That is, for all $i \in N$,

$$v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(\sigma_i, \sigma_{-i}^*) \ \ \forall \sigma_i \in \Delta S_i.$$

Proposition 6.1 Given $\sigma_i \in \Delta S_i$, denote $S_i^+(\sigma_i) = \{s_i \in S_i | \sigma_i(s_i) > 0\}$ (the set of the support of σ_i^*). A mixed strategy profile σ is a Nash equilibrium if and only if $\forall i \in N$

(i) $\forall s_i \in S_i^+(\sigma_i), \forall \hat{s}_i \in S_i^+(\sigma_i), v_i(s_i, \sigma_{-i}) = v_i(\hat{s}_i, \sigma_{-i}),$

(ii) $\forall s_i \in S_i^+(\sigma_i), \forall \hat{s}_i \notin S_i^+(\sigma_i), v_i(s_i, \sigma_{-i}) \geq v_i(\hat{s}_i, \sigma_{-i}).$

Proposition 6.1 Given $\sigma_i \in \Delta S_i$, denote $S_i^+(\sigma_i) = \{s_i \in S_i | \sigma_i(s_i) > 0\}$. A mixed strategy profile σ is a Nash equilibrium if and only if $\forall i \in N$

(i) $\forall s_i \in S_i^+(\sigma_i), \forall \hat{s}_i \in S_i^+(\sigma_i), v_i(s_i, \sigma_{-i}) = v_i(\hat{s}_i, \sigma_{-i})$,

(ii) $\forall s_i \in S_i^+(\sigma_i), \forall \hat{s}_i \notin S_i^+(\sigma_i), v_i(s_i, \sigma_{-i}) \geq v_i(\hat{s}_i, \sigma_{-i})$.

Proof (rough sketch) "only if part": if either (i) or (ii) does not hold for some i, there are strategies $s_i \in S_i^+$ and $s_i' \in S_i$ such that $v_i(s_i', \sigma_{-i}) > v_i(s_i, \sigma_{-i})$.

Proposition 6.1 Given $\sigma_i \in \Delta S_i$, denote $S_i^+(\sigma_i) = \{s_i \in S_i | \sigma_i(s_i) > 0\}$. A mixed strategy profile σ is a Nash equilibrium if and only if $\forall i \in N$

(i) $\forall s_i \in S_i^+(\sigma_i), \forall \hat{s}_i \in S_i^+(\sigma_i), v_i(s_i, \sigma_{-i}) = v_i(\hat{s}_i, \sigma_{-i})$,

(ii) $\forall s_i \in S_i^+(\sigma_i), \forall \hat{s}_i \notin S_i^+(\sigma_i), v_i(s_i, \sigma_{-i}) \geq v_i(\hat{s}_i, \sigma_{-i})$.

Proof (rough sketch) "if part": Suppose that (i) and (ii) hold but σ is not a Nash equilibrium. There is some i who has a strategy σ_i' with $v_i(\sigma_i', \sigma_{-i}) > v_i(\sigma_i, \sigma_{-i})$. For some pure strategy s_i' with $\sigma_i'(s_i') > 0$, $v_i(s_i', \sigma_{-i}) > v_i(\sigma_i, \sigma_{-i})$.

Example (1)

All-pay-auction There are two players who can bid for a dollar. Each can set a bid that is on the interval $[0, 1]$, that is, $S_i = [0, 1]$. The players need to pay their bids, s_1 and s_2, regardless of the bidding outcome.

Example (1)

All-pay-auction There are two players who can bid for a dollar. $S_i = [0,1]$. The players need to pay their bids, s_1 and s_2, regardless of the bidding outcome.

The person with the higher bid gets the dollar. If there is a tie, the dollar is awarded to each with prob 1/2.

The payoff of player i is

$$v_i(s_i, s_{-i}) = \begin{cases} -s_i & \text{if } s_i < s_j \\ 1/2 - s_i & \text{if } s_i = s_j \\ 1 - s_i & \text{if } s_i > s_j. \end{cases}$$

(1) Show that there is no pure strategy Nash equilibrium.

Example (1)

All-pay-auction There are two players who can bid for a dollar. $S_i = [0, 1]$. The players need to pay their bids, s_1 and s_2, regardless of the bidding outcome.

The payoff of player i is

$$v_i(s_i, s_{-i}) = \begin{cases} -s_i & \text{if } s_i < s_j \\ 1/2 - s_i & \text{if } s_i = s_j \\ 1 - s_i & \text{if } s_i > s_j. \end{cases}$$

(2) Show that the following is a Nash equilibrium:

$$F_i(s_i) = s_i \text{ for } s_i \in [0, 1] \ (i = 1, 2).$$

Page 107 in Tadelis shows that $v_i(s_i, \sigma_j) = 0 \ \forall s_i \in [0, 1]$.

Example (2)

Matching Pennies No pure strategy Nash equilibrium.

1/2	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Consider the following mixed strategy profile:
$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q)$.

Example (2)

Matching Pennies No pure strategy Nash equilibrium.

1/2	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

$$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q).$$

Player 1's expected payoff for each pure strategy

$$\begin{aligned} v_1(H, \sigma_2) &= q \times 1 + (1-q) \times (-1) = 2q - 1, \\ v_1(T, \sigma_2) &= q \times (-1) + (1-q) \times 1 = 1 - 2q, \end{aligned}$$

Example (2)

Matching Pennies No pure strategy Nash equilibrium.

$1/2$	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q)$.

Player 1's expected payoff for each pure strategy

$$\begin{aligned} v_1(H, \sigma_2) &= q \times 1 + (1-q) \times (-1) = 2q - 1, \\ v_1(T, \sigma_2) &= q \times (-1) + (1-q) \times 1 = 1 - 2q, \end{aligned}$$

Playing H (T) is strictly better iff $q > 1/2$ ($q < 1/2$).
Each of them is indifferent iff $q = 1/2$.

Example (2)

Matching Pennies No pure strategy Nash equilibrium.

$1/2$	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q)$.

Playing H (T) is strictly better iff $q > 1/2$ ($q < 1/2$).
Each of them is indifferent iff $q = 1/2$.

The best-response correspondence of player 1

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 1 & \text{if } q > 1/2. \end{cases}$$

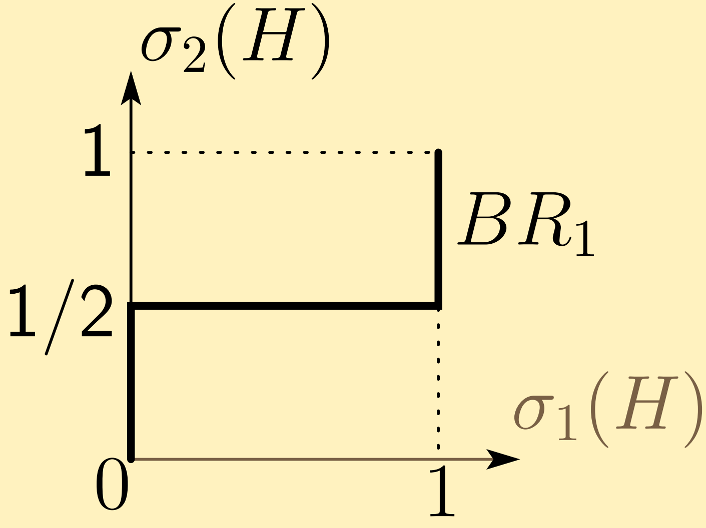
Example (2)

Matching Pennies No pure strategy Nash equilibrium.

$1/2$	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

$$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q).$$

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 1 & \text{if } q > 1/2. \end{cases}$$



Example (2)

Matching Pennies No pure strategy Nash equilibrium.

1/2	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

$$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q).$$

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 1 & \text{if } q > 1/2. \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < 1/2 \\ q \in [0,1] & \text{if } p = 1/2 \\ q = 0 & \text{if } p > 1/2. \end{cases}$$

Example (2)

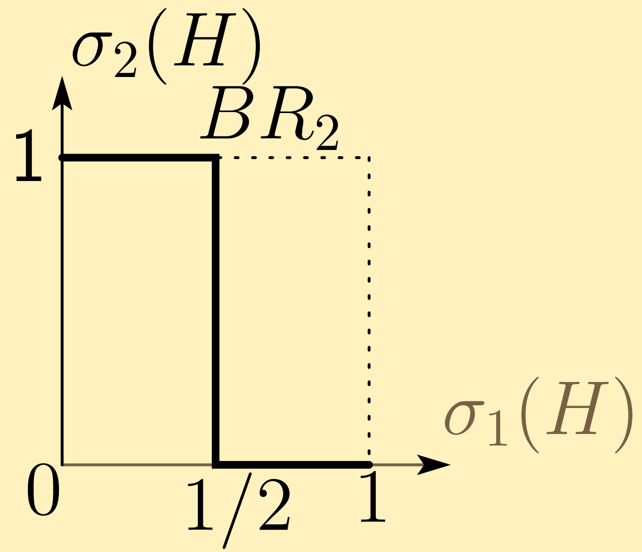
Matching Pennies No pure strategy Nash equilibrium.

1/2	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

$$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q).$$

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 1 & \text{if } q > 1/2. \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < 1/2 \\ q \in [0,1] & \text{if } p = 1/2 \\ q = 0 & \text{if } p > 1/2. \end{cases}$$



Example (2)

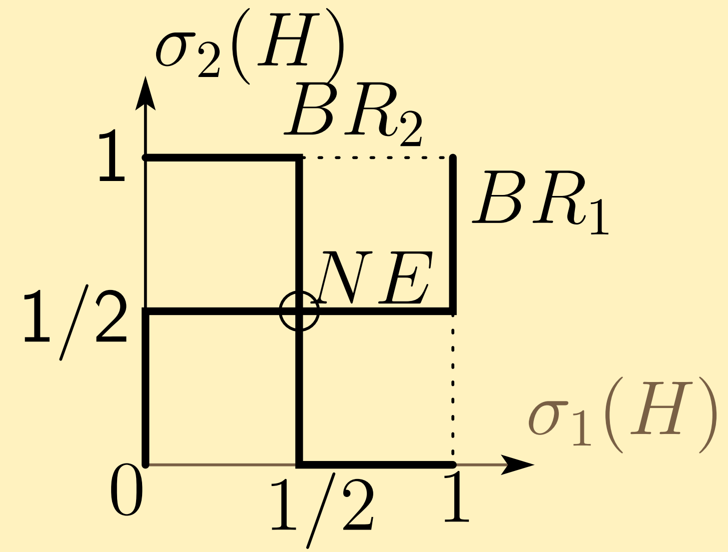
Matching Pennies No pure strategy Nash equilibrium.

1/2	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

$$\sigma = (\sigma_1(H), \sigma_1(T), \sigma_2(H), \sigma_2(T)) = (p, 1-p, q, 1-q).$$

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < 1/2 \\ p \in [0,1] & \text{if } q = 1/2 \\ p = 1 & \text{if } q > 1/2. \end{cases}$$

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < 1/2 \\ q \in [0,1] & \text{if } p = 1/2 \\ q = 0 & \text{if } p > 1/2. \end{cases}$$



Example (2)

Matching Pennies No pure strategy Nash equilibrium.

$1/2$	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

- If a player is mixing several strategies, he must be *indifferent between them*.
- In the Matching Pennies game, we check *which strategy of player 2* will make player 1 indifferent between playing H and T.

Example (3)

Rock-paper-Scissors No pure strategy Nash equilibrium.

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Example (3)

Rock-paper-Scissors No pure strategy Nash equilibrium.

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

No Nash eq. where one player plays a pure strategy and the other mixes.

Example (3)

Rock-paper-Scissors No pure strategy Nash equilibrium.

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

No Nash eq. where at least one player mixes only two of the pure strategies.
By symmetry, suppose that player i mixes only R and P.
Given this, $P \succ_j R$, which induces player j not to play R with a positive prob.. Given this response by player j, $S \succ_i P$, which changes the initially assumed strategy of player i. A contradiction.

Example (3)

Rock-paper-Scissors No pure strategy Nash equilibrium.

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Each player must mix the three strategies.

Suppose that player i's mixed strategy is

$$\sigma_i = (\sigma_i(R), \sigma_i(P), 1 - \sigma_i(R) - \sigma_i(P)).$$

Example (3)

Rock-paper-Scissors No pure strategy Nash equilibrium.

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Each player must mix the three strategies.

Suppose that player i's mixed strategy is

$$\sigma_i = (\sigma_i(R), \sigma_i(P), 1 - \sigma_i(R) - \sigma_i(P)).$$

$v_j(R, \sigma_i) = 1 - \sigma_i(R) - 2\sigma_i(P),$

$v_j(P, \sigma_i) = -1 + 2\sigma_i(R) + \sigma_i(P).$

$v_j(S, \sigma_i) = -\sigma_i(R) + \sigma_i(P).$

Example (3)

Rock-paper-Scissors No pure strategy Nash equilibrium.

1/2	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Each player must mix the three strategies.

Suppose that player i's mixed strategy is

$$\sigma_i = (\sigma_i(R), \sigma_i(P), 1 - \sigma_i(R) - \sigma_i(P)).$$

$$v_j(R, \sigma_i) = 1 - \sigma_i(R) - 2\sigma_i(P),$$

$$v_j(P, \sigma_i) = -1 + 2\sigma_i(R) + \sigma_i(P).$$

$$v_j(S, \sigma_i) = -\sigma_i(R) + \sigma_i(P).$$

If $v_j(R, \sigma_i) = v_j(P, \sigma_i) = v_j(S, \sigma_i)$, player j mixes R, P, and S.

Example (4)

Bertrand competition (Kwong, 2003, CJE) Two firms with identical constant marginal cost c exist.

Example (4)

Bertrand competition (Kwong, 2003, CJE) Two firms with identical constant marginal cost c exist.

Consumers Type i loyal consumers buy only product i iff $p_i \leq r$ (r is their reservation price). The mass is L.

Example (4)

Bertrand competition (Kwong, 2003, CJE) Two firms with identical constant marginal cost c exist.

Consumers Type i loyal consumers buy only product i iff $p_i \leq r$ (r is their reservation price). The mass is L. Type 0 consumers buy a product with the lowest price, if which is not larger than r. The mass is M.

Example (4)

Consumers Type i loyal consumers buy only product i iff $p_i \leq r$ (r is their reservation price). The mass is L. Type 0 consumers buy a product with the lowest price, if which is not larger than r. The mass is M.

Profits The profit of firm i is given by $(i = 1, 2)$

$$\Pi^i(p_i, p_j) = \begin{cases} (p_i - c)(M + L) & \text{if } p_i < p_j, \\ (p_i - c)(M/2 + L) & \text{if } p_i = p_j, \\ (p_i - c)L & \text{if } p_i > p_j. \end{cases}$$

Example (4)

Profits The profit of firm i is given by $(i = 1, 2)$

$$\Pi^i(p_i, p_j) = \begin{cases} (p_i - c)(M + L) & \text{if } p_i < p_j, \\ (p_i - c)(M/2 + L) & \text{if } p_i = p_j, \\ (p_i - c)L & \text{if } p_i > p_j. \end{cases}$$

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$. Assume that F_i is continuously differentiable and that f_i denotes the density function.

Example (4)

Profits The profit of firm i is given by $(i = 1, 2)$

$$\Pi^i(p_i, p_j) = \begin{cases} (p_i - c)(M + L) & \text{if } \; p_i < p_j, \\ (p_i - c)(M/2 + L) & \text{if } \; p_i = p_j, \\ (p_i - c)L & \text{if } \; p_i > p_j. \end{cases}$$

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.

Prove that the support of the price density is given by $[\underline{p}, r]$.

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.

Prove that the support of the price density is given by $[\underline{p}, r]$.

- For any $p_i(> r)$, firm i earns no profit.

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.

Prove that the support of the price density is given by $[\underline{p}, r]$.

- For any $p_i(> r)$, firm i earns no profit.
- Each firm can earn a profit at least $(r - c)L$.

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.

Prove that the support of the price density is given by $[\underline{p}, r]$.

- For any $p_i(> r)$, firm i earns no profit.
- Each firm can earn a profit at least $(r - c)L$.
- A critical price below which it is unprofitable to price.

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.

Prove that the support of the price density is given by $[\underline{p}, r]$.

- For any $p_i(> r)$, firm i earns no profit.
- Each firm can earn a profit at least $(r - c)L$.
- A critical price below which it is unprofitable to price. This critical price, $\underline{p}$, is derived by the equation, $(\underline{p} - c)(M + L) = (r - c)L$.

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.
The expected profit of firm i is given as

$$\int_{\underline{p}}^{r} \left[(1 - F_j(p_i))\, (p_i - c)(M + L) + F_j(p_i)(p_i - c)L \right] f_i(p_i) dp_i,$$

Solution Applying Proposition 6.1, we have the following equation:

$$(1 - F_j(p_i))\, (p_i - c)(M + L) + F_j(p_i)(p_i - c)L = (r - c)L.$$

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.
The expected profit of firm i is given as

$$\int_{\underline{p}}^{r} \left[(1 - F_j(p_i))\,(p_i - c)(M + L) + F_j(p_i)(p_i - c)L\right] f_i(p_i)dp_i,$$

Solution Applying Proposition 6.1, we have the following equation:

$$(1 - F_j(p_i))\,(p_i - c)(M + L) + F_j(p_i)(p_i - c)L = (r - c)L.$$

Solving the equation with respect to F, we have

$$F_i(p_i) = 1 - \frac{(r - p_i)L}{(p_i - c)M}.$$

Example (4)

Mixed strategy Firm i randomizes p_i according to some distribution function $F_i(p_i)$.
The expected profit of firm i is given as

$$\int_{\underline{p}}^{r} \left[(1 - F_j(p_i))\,(p_i - c)(M + L) + F_j(p_i)(p_i - c)L \right] f_i(p_i) dp_i,$$

Solution Applying Proposition 6.1, we have the following equation:

$$(1 - F_j(p_i))\,(p_i - c)(M + L) + F_j(p_i)(p_i - c)L = (r - c)L.$$

Solving the equation with respect to F, we have

$$F_i(p_i) = 1 - \frac{(r - p_i)L}{(p_i - c)M}.$$

A related paper (Baye and Morgan, 2001, AER).

Section 6.4

Proposition Every game $\Gamma = [N, \{\Delta S_i\}, \{v_i\}]$ in which the sets $S_1, \ldots, S_n$ have a finite number of elements has a mixed strategy Nash equilibrium.

Proposition A Nash equilibrium exists in game $\Gamma = [N, \{S_i\}, \{v_i\}]$ if $\forall i \in N$,

1. S_i is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$.
2. $v_i(s_1, \ldots, s_n)$ is continuous in $(s_1, \ldots, s_n)$ and quasi-concave in s_i.