Chapter 8: Credibility and Sequential Rationality

Outline

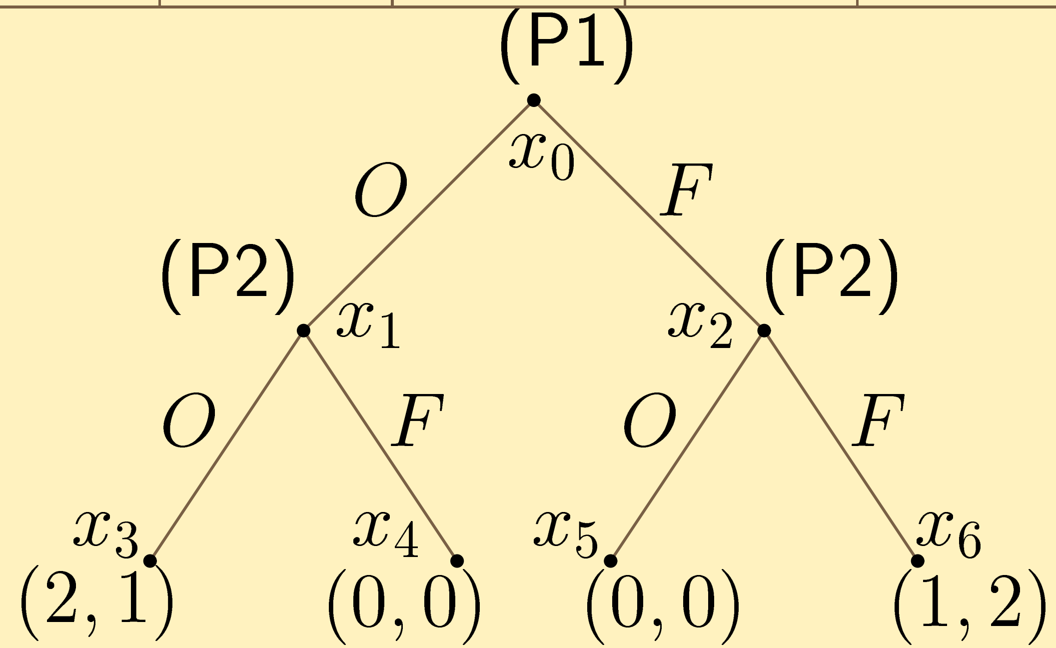
- Backward induction, Zermelo's Theorem.
- Subgame, Subgame perfect Nash equilibrium.
- Reduced game.
- Example.

Backward Induction

$1/2$	O	F
O	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

Example (Battle of the Sexes: Sequential move case)

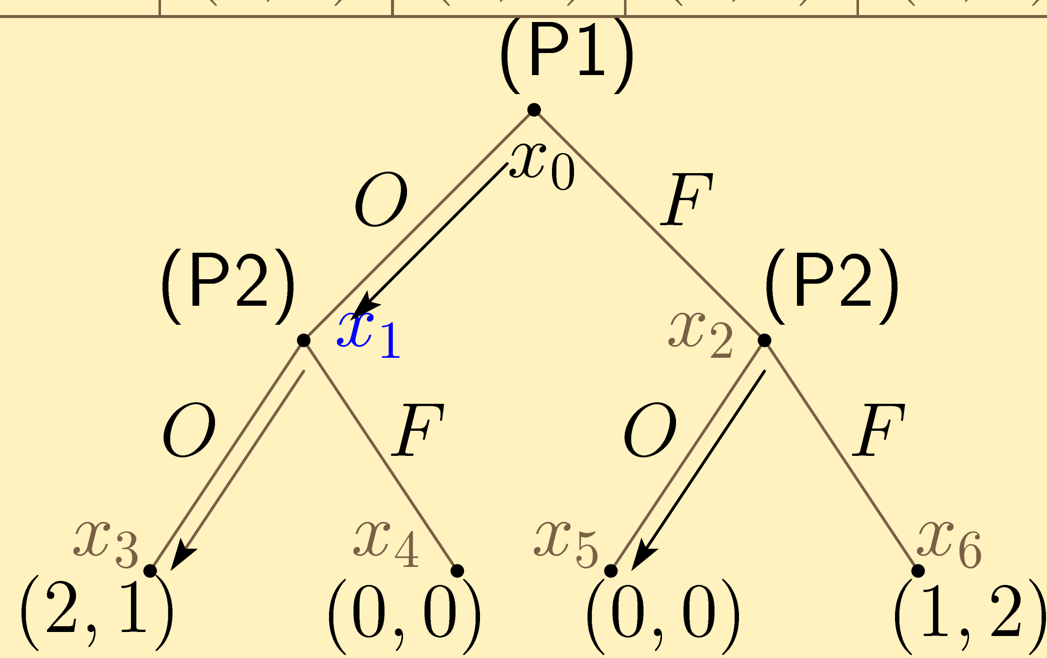
1 / 2	O, O	O, F	F, O	F, F
Opera	$(2, 1)$	$(2, 1)$	$(0, 0)$	$(0, 0)$
Football	$(0, 0)$	$(1, 2)$	$(0, 0)$	$(1, 2)$



There are three Nash equilibria.

Example (Battle of the Sexes: Sequential move case)

1 / 2	O, O	O, F	F, O	F, F
Opera	$(\underline{2}, \bar{1})$	$(\underline{2}, \bar{1})$	$(\underline{0}, 0)$	$(0, 0)$
Football	$(0, 0)$	$(1, \bar{2})$	$(\underline{0}, 0)$	$(\underline{1}, \bar{2})$

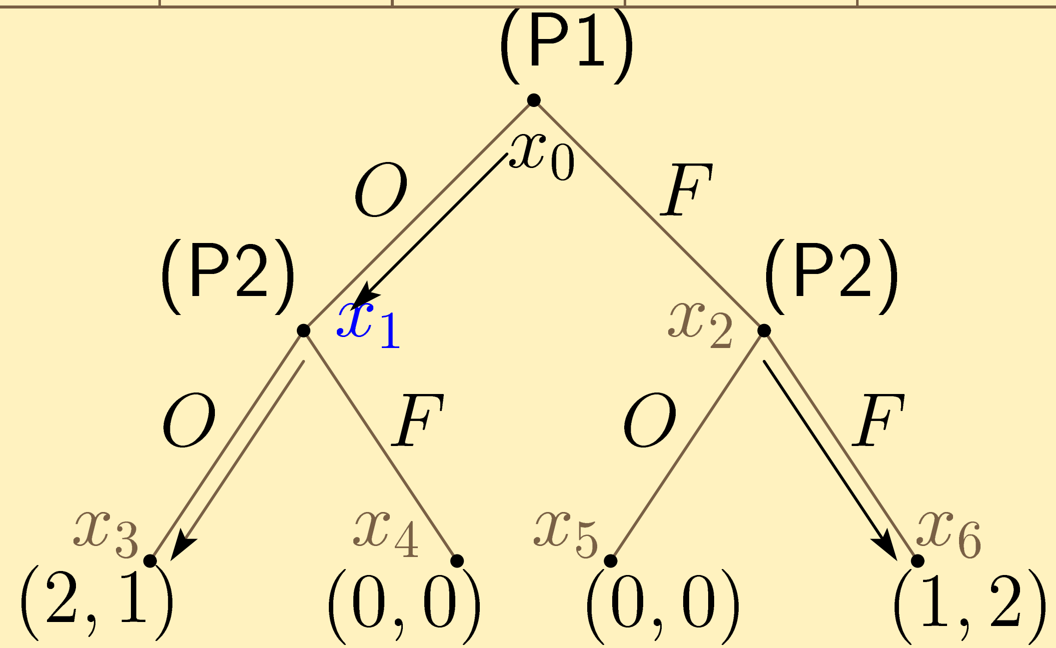


Player 1: O; Player 2: O, O.

Backward Induction

Example (Battle of the Sexes: Sequential move case)

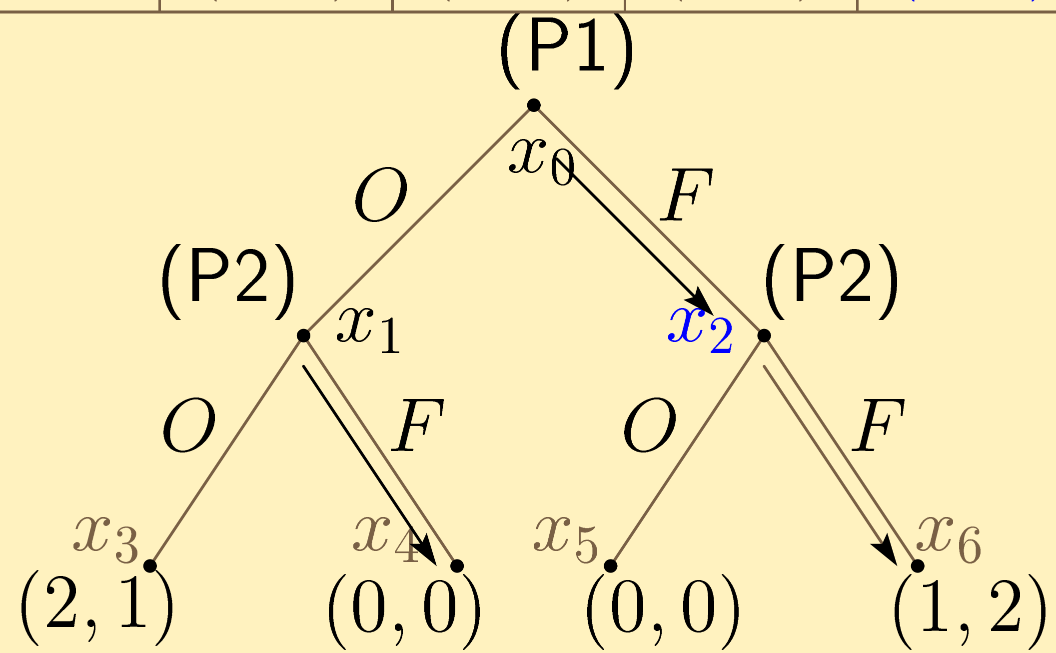
1 / 2	O, O	O, F	F, O	F, F
Opera	$(\underline{2}, \bar{1})$	$(\underline{2}, \bar{1})$	$(\underline{0}, 0)$	$(0, 0)$
Football	$(0, 0)$	$(1, \bar{2})$	$(\underline{0}, 0)$	$(\underline{1}, \bar{2})$



Player 1: O; Player 2: O, F.

Example (Battle of the Sexes: Sequential move case)

1 / 2	O, O	O, F	F, O	F, F
Opera	$(\underline{2}, \bar{1})$	$(\underline{2}, \bar{1})$	$(\underline{0}, 0)$	$(0, 0)$
Football	$(0, 0)$	$(1, \bar{2})$	$(\underline{0}, 0)$	$(\underline{1}, \bar{2})$



Player 1: F; Player 2: F, F.

Backward Induction

Principle of Sequential Rationality A player's strategy should specify optimal action **at every point in the game tree**.

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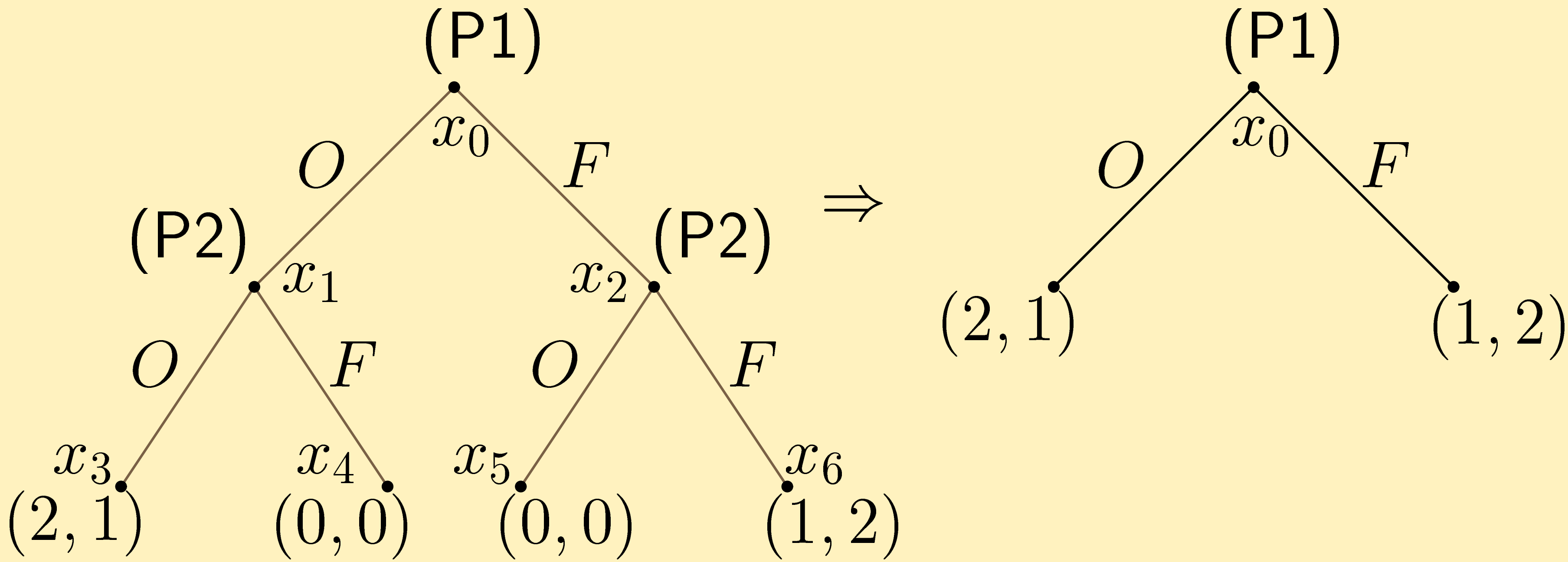
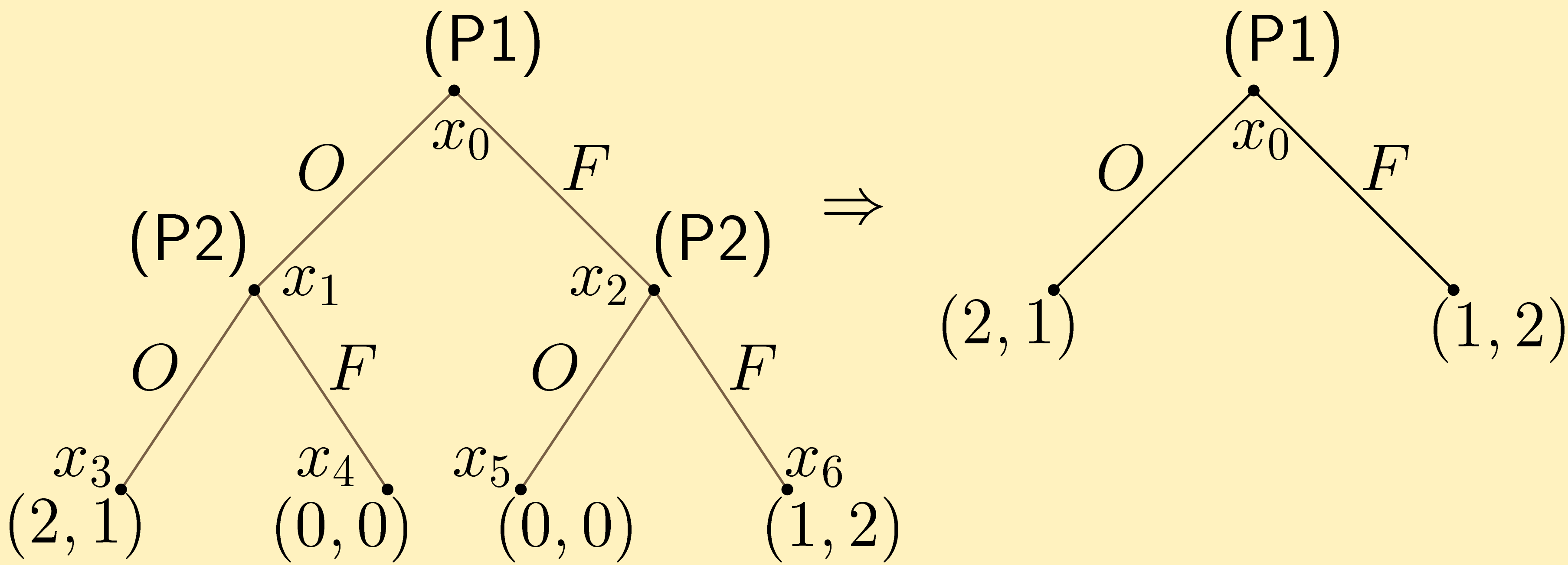
Reduced extensive form game

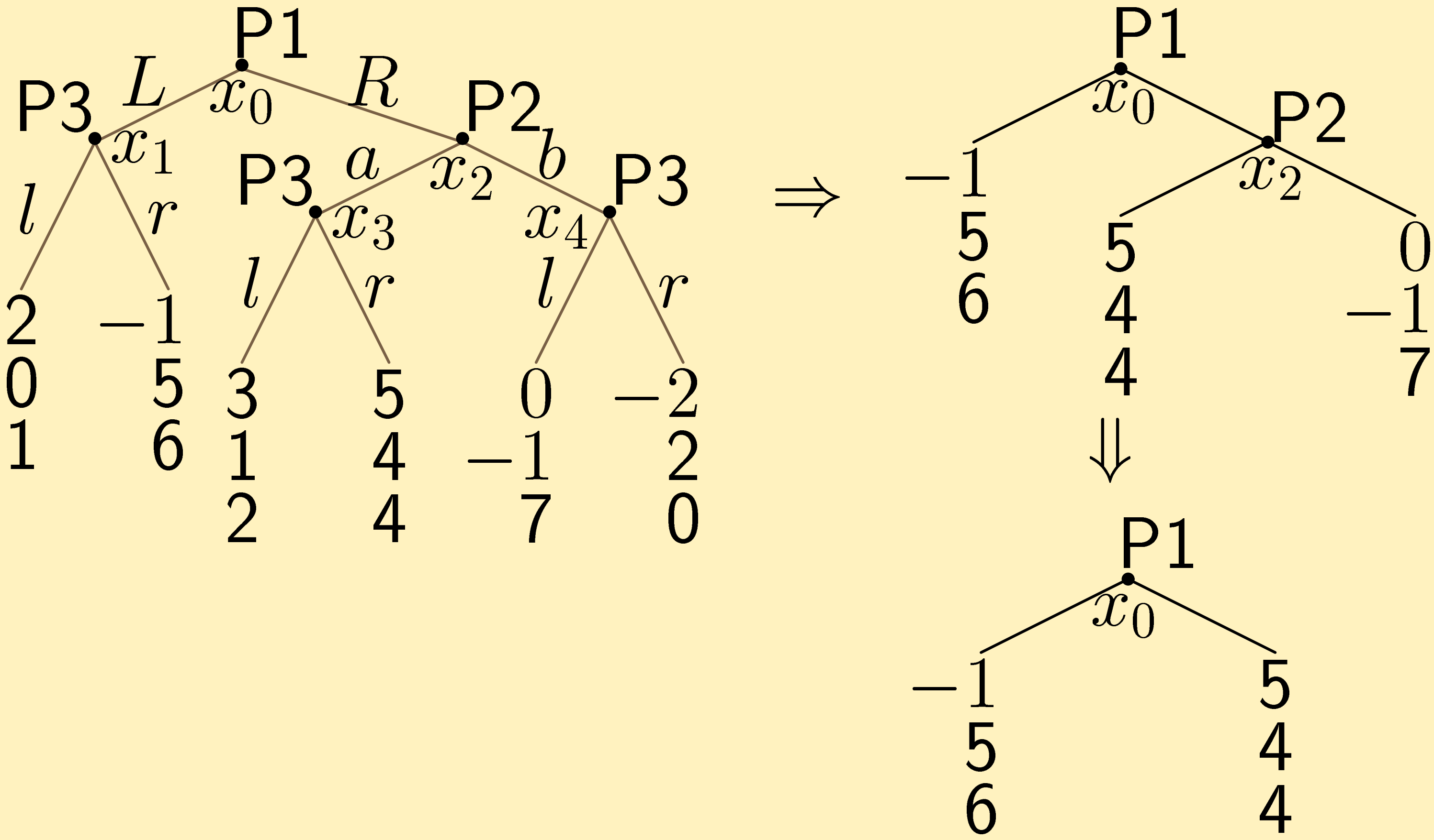


Reduced extensive form game



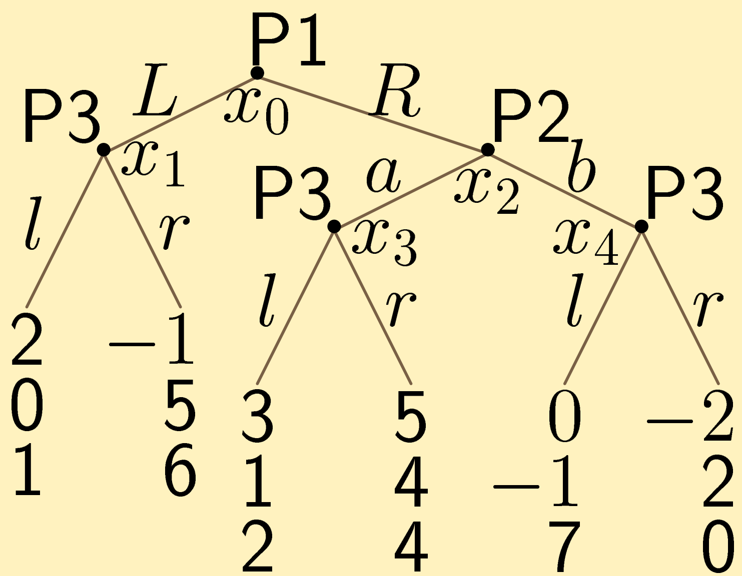
Backward Induction First, solve optimal actions at the final decision nodes. Then, solve optimal actions at the next-to-last decision nodes, and so on.

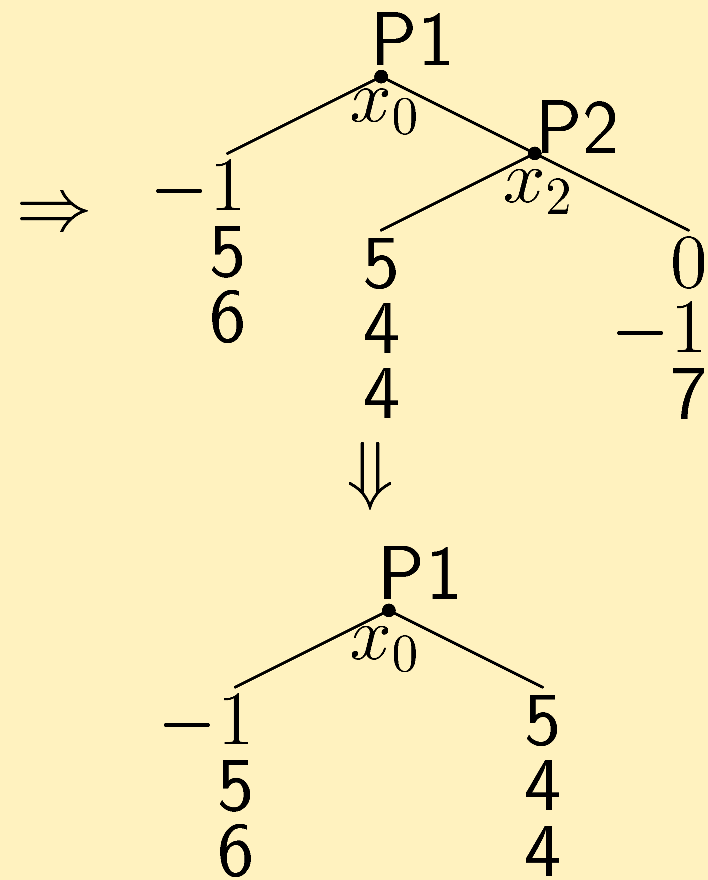
Example (Backward Induction)



Backward Induction

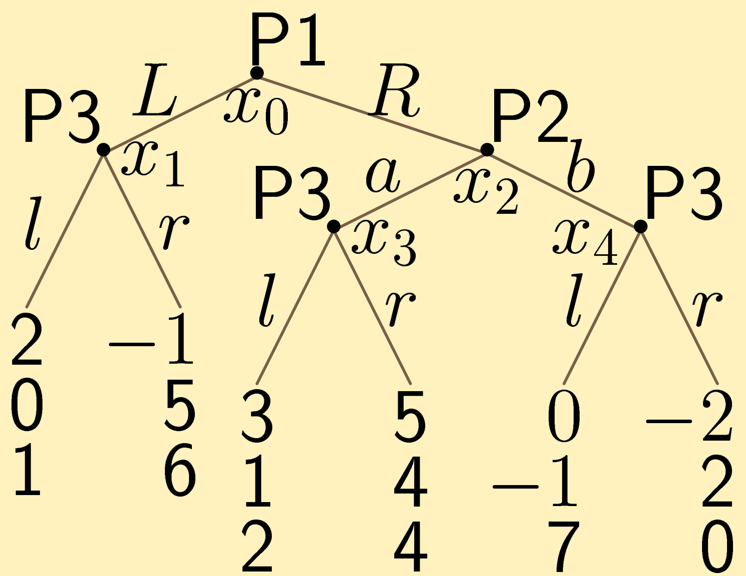
Example (Backward Induction)

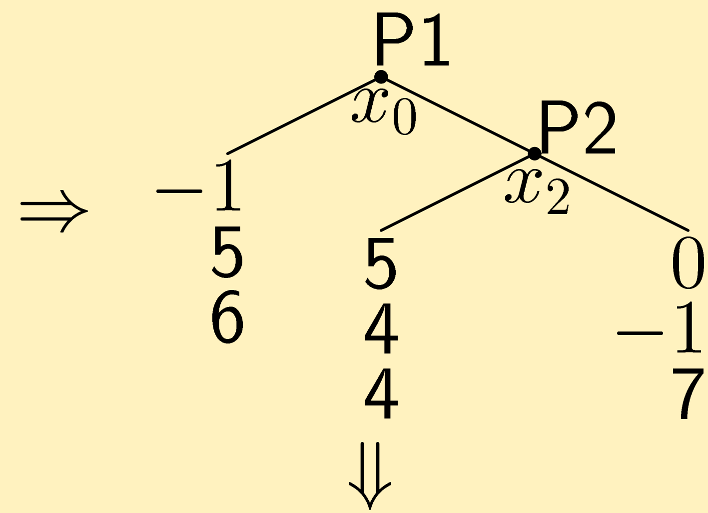


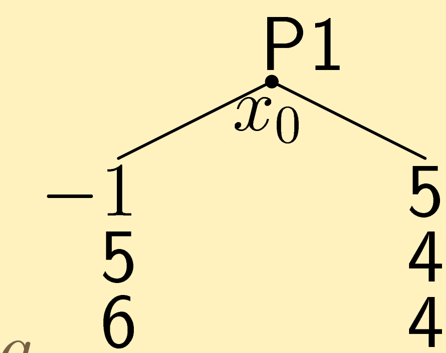


$\sigma_1 = R$, $\sigma_2 =$ "a if P1 plays R"

Example (Backward Induction)







$\sigma_1 = R$, $\sigma_2 =$ "a if P1 plays R"

$$\sigma_3 = \begin{cases} r \text{ if P1 plays } L \\ r \text{ if P1 plays } R \text{ and P2 plays } a \\ l \text{ if P1 plays } R \text{ and P2 plays } b \end{cases}$$

Zermelo's Theorem

Proposition 8.1 (Zermelo's Theorem) (1) Every finite game of **perfect information** Γ has a pure strategy Nash equilibrium that can be derived through backward induction.

Zermelo's Theorem

Proposition 8.1 (Zermelo's Theorem) (1) Every finite game of **perfect information** Γ has a pure strategy Nash equilibrium that can be derived through backward induction.

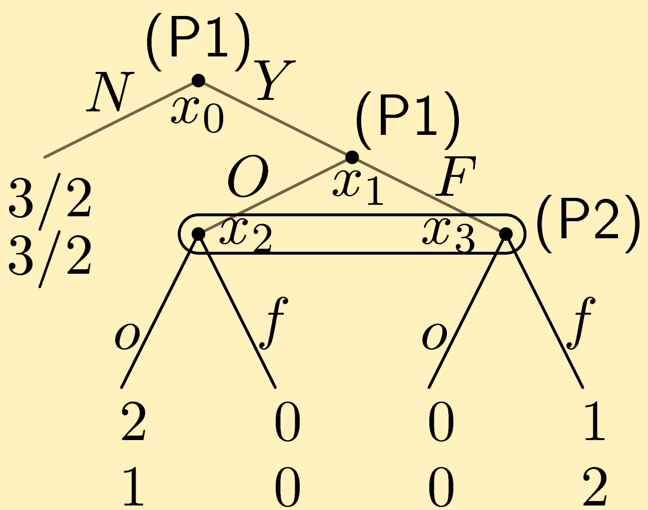
(2) Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

See the supplemental material and MWG pp.272-3 for a formal proof.

See also Schwalbe and Walker (2001, GEB) for the history of this research.

Imperfect information

An extensive form with imperfect information



(P1)/(P2)	o	f
Y, O	$\mathbf{2, 1}$	$0, 0$
Y, F	$0, 0$	$1, 2$
N, O	$3/2, 3/2$	$\mathbf{3/2, 3/2}$
N, F	$3/2, 3/2$	$\mathbf{3/2, 3/2}$

We cannot naively apply the method of backward induction (used previously) to games with imperfect information.

Subgame

Subgame A **subgame** of an extensive form game Γ is a subset of the game having the following properties:

1. It begins with an information set containing a single decision node, contains all the decision nodes that are successors of this node, and contains only these nodes.
2. If decision node x is in the subgame, then every $x' \in h(x)$ is also, where $h(x)$ is the information set that contains decision node x.

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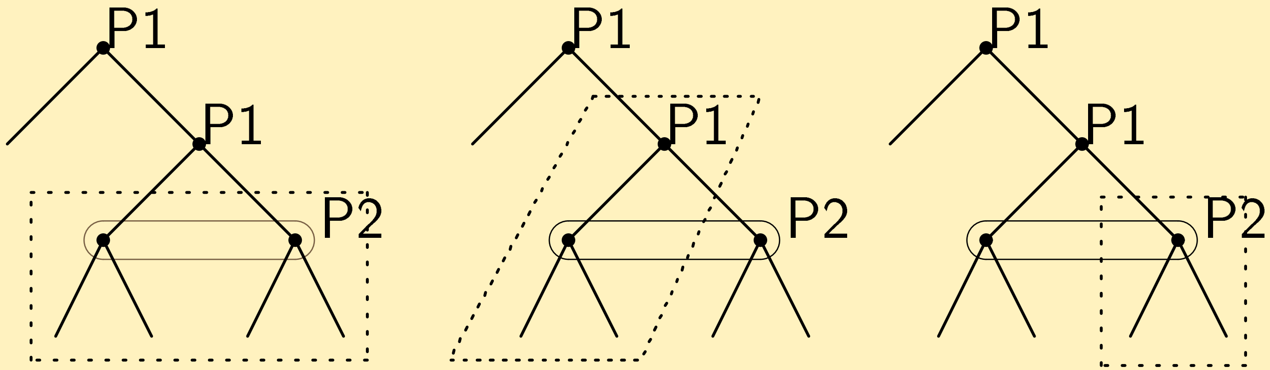

The areas enclosed with dotted lines are NOT subgames.

Subgame perfection

SPNE A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ in an n-player extensive form game Γ is a **subgame perfect Nash equilibrium (SPNE)** if it induces a Nash equilibrium in every subgame of Γ.

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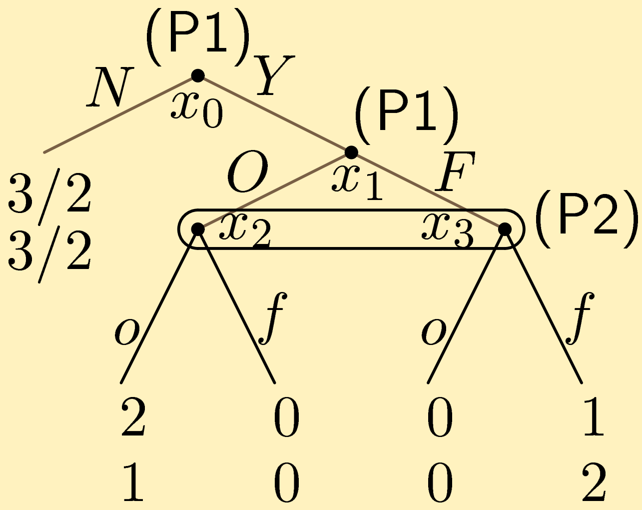
Proposition 8.2 (1) Every finite game of perfect information Γ has a pure strategy subgame perfect Nash equilibrium.

(2) Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.

An extensive form with imperfect information



P1/P2	o	f
Y, O	$\mathbf{2}, \mathbf{1}$	$0, 0$
Y, F	$0, 0$	$1, 2$
N, O	$3/2, 3/2$	$\mathbf{3/2}, \mathbf{3/2}$
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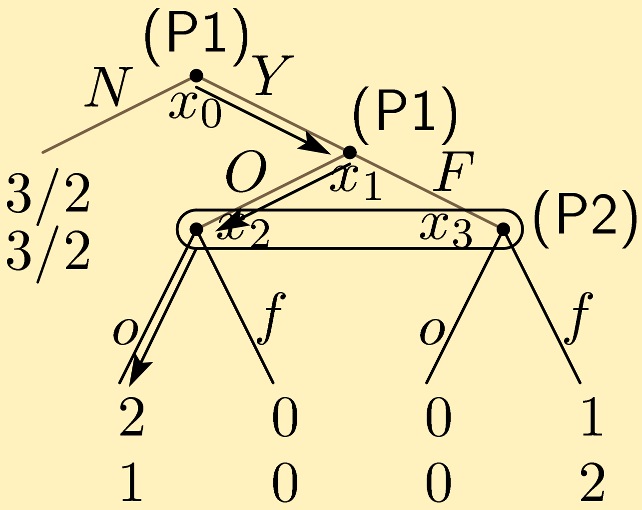
An extensive form with imperfect information



$((Y, O), o)$

P1/P2	o	f
Y, O	$\mathbf{2, 1}$	$0, 0$
Y, F	$0, 0$	$1, 2$
N, O	$3/2, 3/2$	$\mathbf{3/2, 3/2}$
N, F	$3/2, 3/2$	$\mathbf{3/2, 3/2}$

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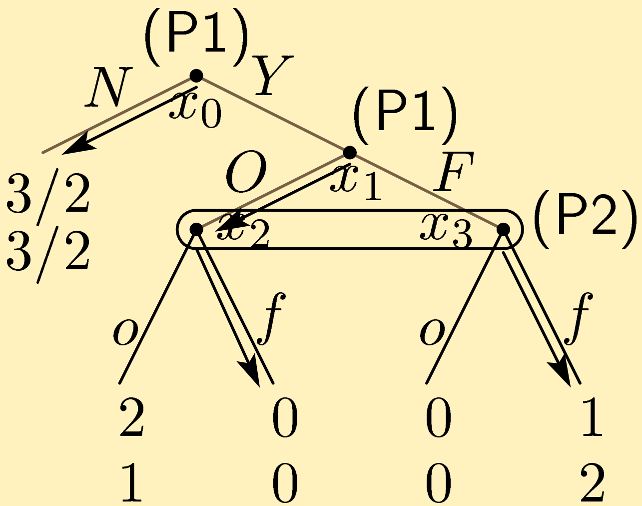

$((N, O), f)$

(unreasonable)

P1/P2	o	f
Y, O	$\mathbf{2}, \mathbf{1}$	$0, 0$
Y, F	$0, 0$	$1, 2$
N, O	$3/2, 3/2$	$\mathbf{3/2}, \mathbf{3/2}$
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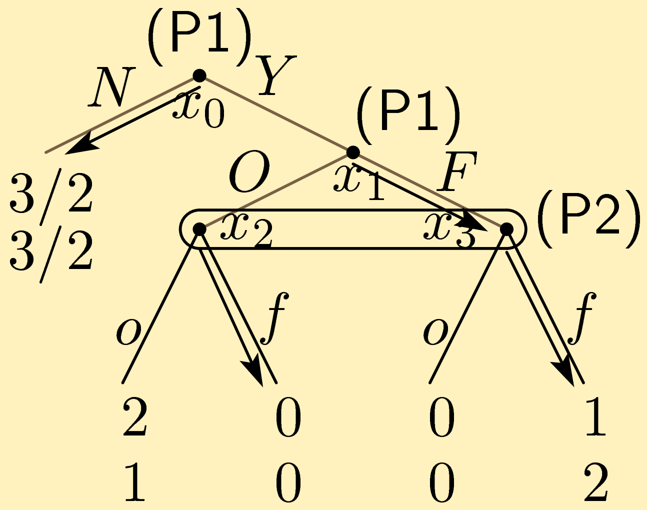
An extensive form with imperfect information



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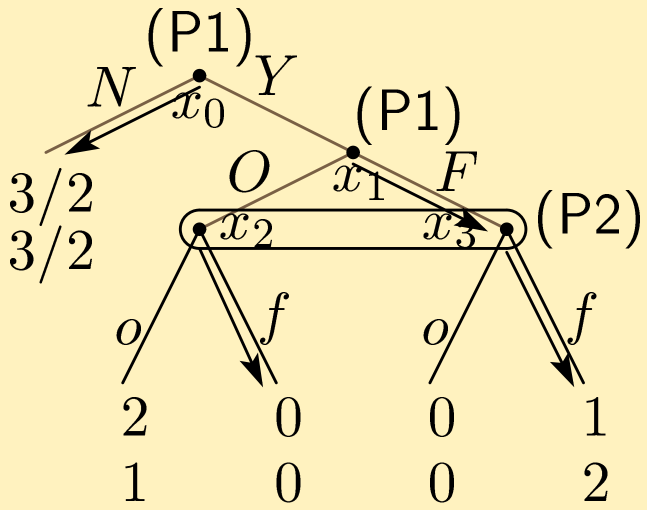
$((N, F), f)$

An extensive form with imperfect information



$((N, F), f)$

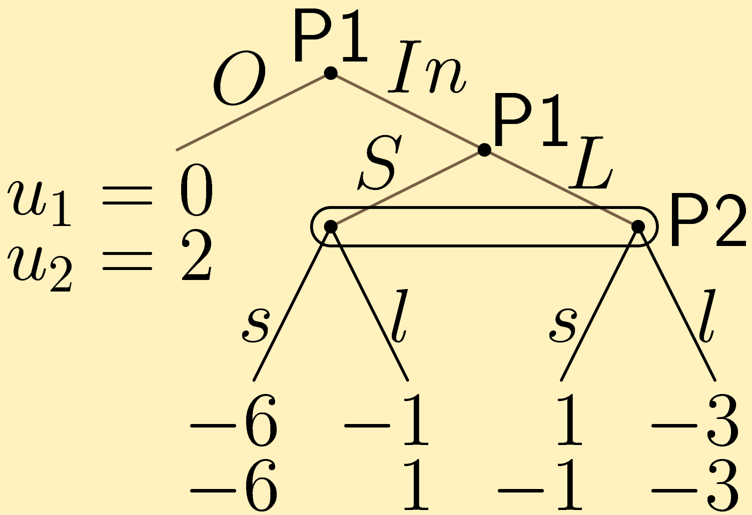
P1/P2	o	f
Y, O	$\mathbf{2, 1}$	$0, 0$
Y, F	$0, 0$	$1, 2$
N, O	$3/2, 3/2$	$\mathbf{3/2, 3/2}$
N, F	$3/2, 3/2$	$\mathbf{3/2, 3/2}$

Two SPNE exist: $((Y, O), o)$ and $((N, F), f)$.

Multiple equilibria

Subgame

P1/P2	s	l
S	$-6, -6$	$\mathbf{-1, 1}$
L	$\mathbf{1, -1}$	$-3, -3$



Multiple equilibria

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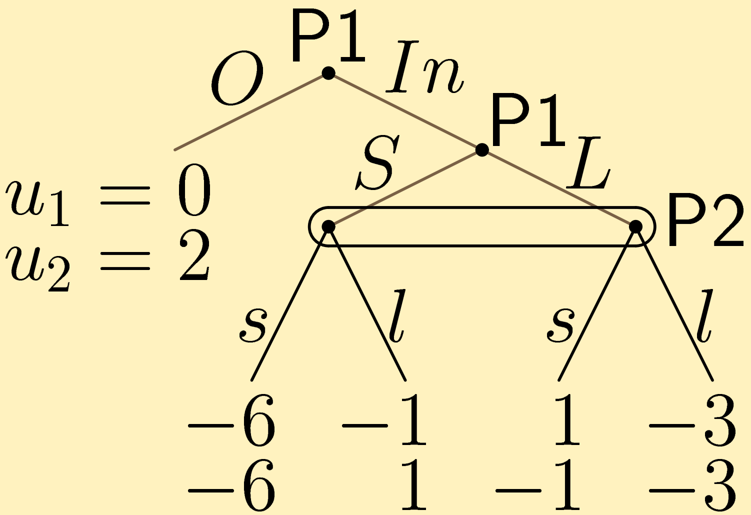

The subgame has two Nash equilibria: (L, s) and (S, l).

Multiple equilibria

Subgame

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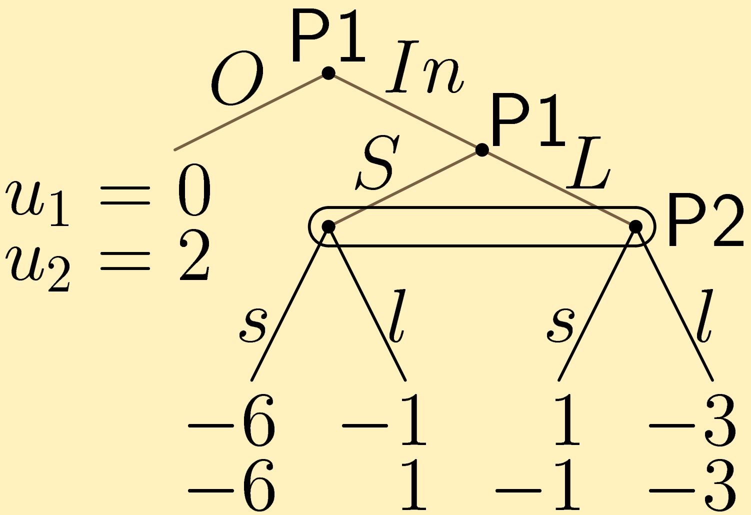
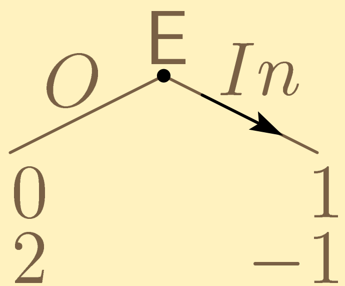
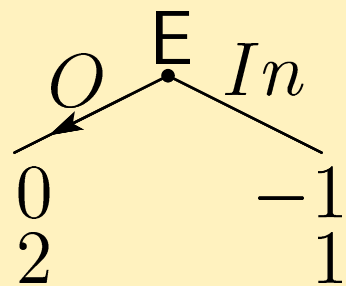


The subgame has two Nash equilibria: (L, s) and (S, l).



(L, s)



(S, l)

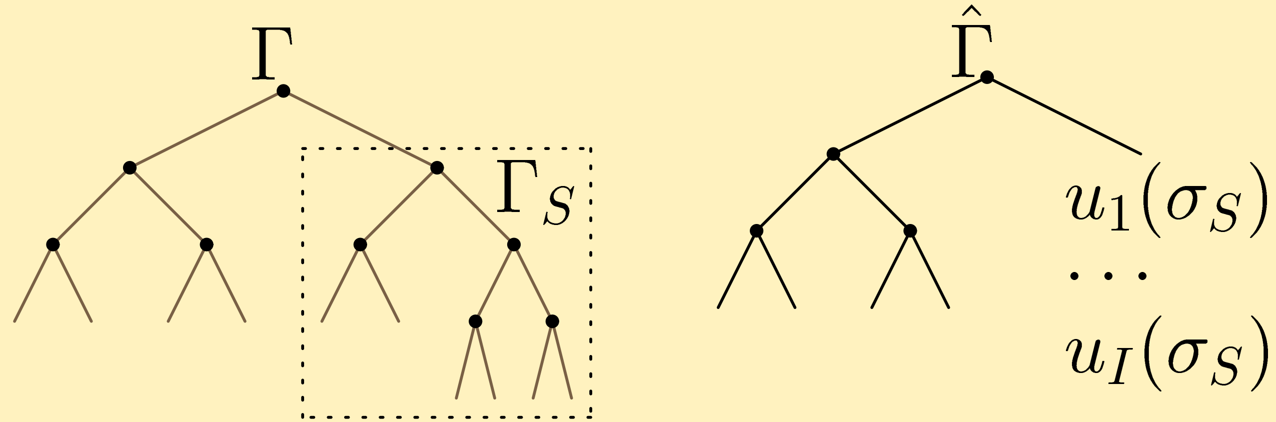
Two SPNE: $((In, L), s)$, $((Out, S), l)$

Generalized Backward Induction

Generalized Backward Induction (MWG)

1. Start at the final subgames, and identify Nash equilibria of them.
2. Select one Nash equilibrium in each of the final subgames, and derive the reduced game in which the final subgames are replaced by the payoffs that result in the subgames when players use the selected Nash equilibrium strategies.
3. Repeat Steps 1 & 2 until every move in Γ is determined.

Illustration of the next slide



(i) $\sigma = (\sigma_S, \hat{\sigma})$ is a SPNE of $\Gamma \Rightarrow \hat{\sigma}$ is a SPNE of $\hat{\Gamma}$.
(ii) $\hat{\sigma}$ is a SPNE of $\hat{\Gamma} \Rightarrow \sigma = (\sigma_S, \hat{\sigma})$ is a SPNE of Γ.

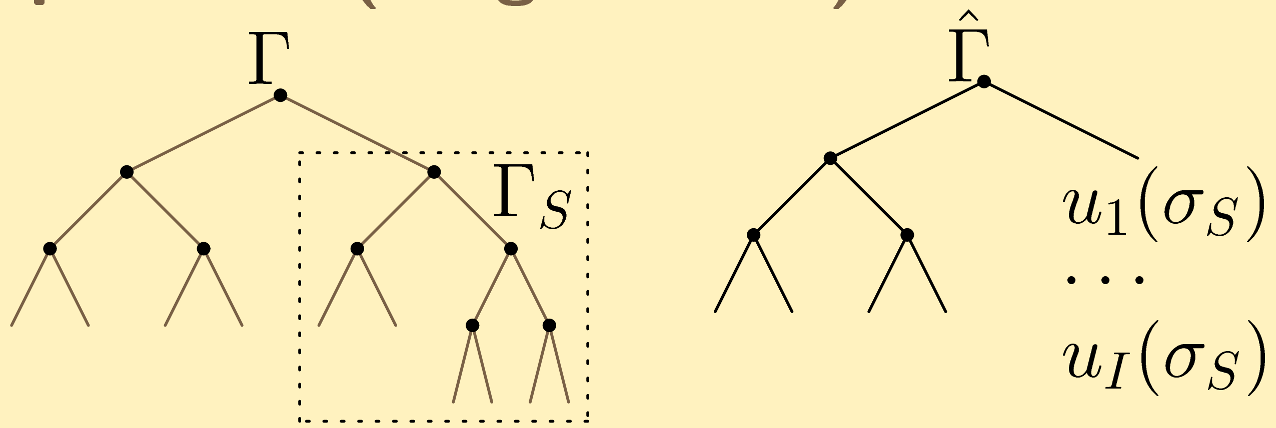
Reduced game

Proposition 9.B.3 (MWG) Let Γ_S be a subgame of Γ. Let the strategy profile σ_S be a SPNE of Γ_S, and let $\hat{\Gamma}$ be the reduced game formed by replacing subgame Γ_S by a terminal node with payoffs equal to those arising from σ_S.

(i) In any SPNE σ of Γ in which σ_S is the play in Γ_S, player's move at information sets outside Γ_S constitutes a SPNE of $\hat{\Gamma}$.

(ii) If $\hat{\sigma}$ is a SPNE of $\hat{\Gamma}$, then the strategy profile σ that specifies the moves in σ_S at information sets in Γ_S and that specifies the moves in $\hat{\sigma}$ at information sets not in Γ_S is a SPNE of Γ.

Proof of Prop. 9.B.3 (rough sketch)



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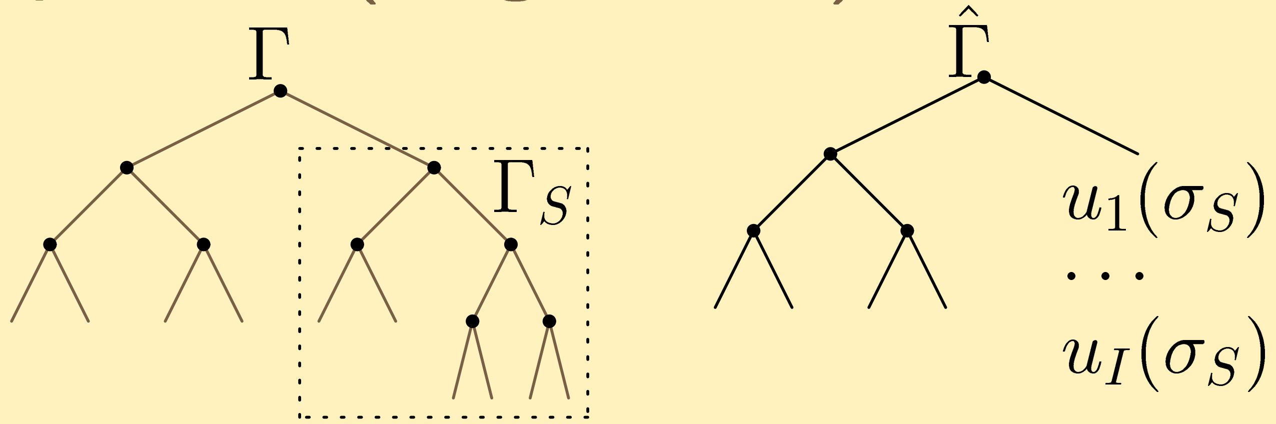


(i) $\sigma = (\sigma_S, \hat{\sigma})$ is a SPNE of Γ $\Rightarrow$ $\hat{\sigma}$ is a SPNE of $\hat{\Gamma}$.

Suppose that $\hat{\sigma}$ is NOT a SPNE of $\hat{\Gamma}$.

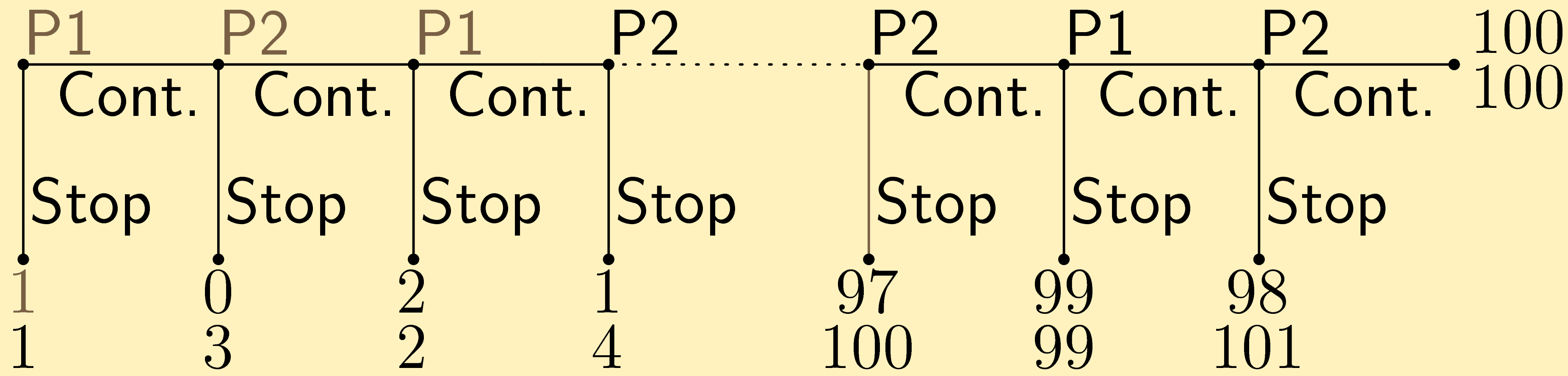
There is a subgame of $\hat{\Gamma}$ where $\hat{\sigma}$ does not constitute a NE.

Some player has an incentive to deviate in the subgame.

She also deviates in the corresponding subgame of Γ.

Her moves do not change in S. σ is NOT a SPNE of Γ.

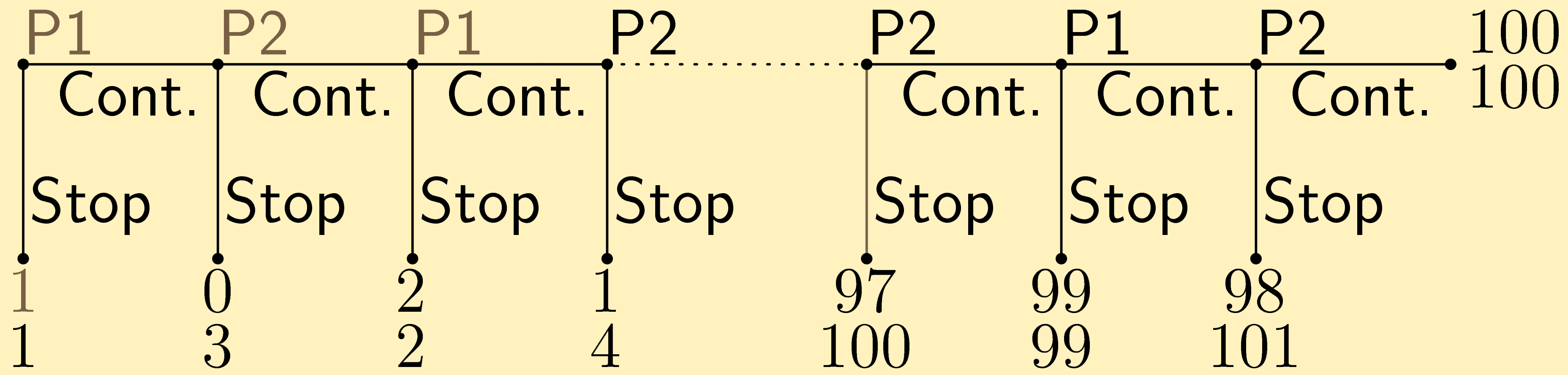
Centipede Game



Each player chooses ‘Stop’ at *all* information sets.

Example

Centipede Game



Each player chooses 'Stop' at *all* information sets.

The backward induction solution is *unappealing* (Rosenthal, 1981, *Journal of Economic Theory*).

There are many experimental studies on the game (e.g., Palacios-Huerta and Volij (2009, *AER*) and Kawagoe and Takizawa (2012, *J. Economic Behavior and Organization*)).

Chapter 9: Multistage Games

Outline

- Multistage game (1)
  There is a unique Nash equilibrium in each stage.
- Multistage game (2)
  There are Nash equilibria in stage 2 in a two period model.

Multistage games (1)

Proposition 9.B.4 (MWG) Let Γ involve successive play of T simultaneous move games ("stage-games"), $\Gamma^t = [N, \{\Delta(S_i^t)\}, \{v_i^t(\cdot)\}]$ for $t = 1, \ldots, T$, with players observing the pure strategy played in each game immediately after its play is concluded.

Multistage games (1)

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Assume if s^t is played for each t, then for all $i \in N$, player i's payoff of Γ is $\sum_{t=1}^{T} v_i^t(s^t)$.

Multistage games (1)

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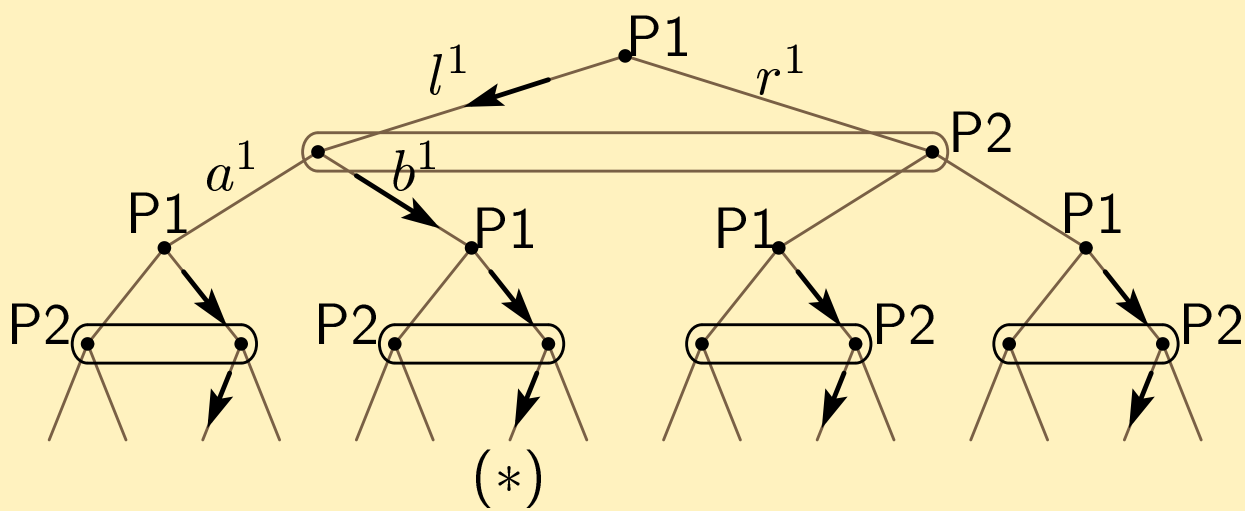
Assume if s^t is played for each t, then for all $i \in N$, player i's payoff of Γ is $\sum_{t=1}^{T} v_i^t(s^t)$.

If there is a **unique** Nash equilibrium $\sigma^{t*} = (\sigma_1^{t\,*}, \ldots, \sigma_N^{t\,*})$ of Γ^t for each $t = 1, \ldots, T$, then there is a **unique** SPNE of Γ, and it consists of each player i playing strategy σ_i^{t*} in Γ^t for each t regardless of what happens previously.

Multistage games (1)

Illustration of this Proposition $(n = 2,\ t = 1, 2)$

$$\Gamma^t = [N, \{\Delta(S_i^t)\}, \{v_i^t(\cdot)\}],\ S_1^t = \{l^t, r^t\},\ S_2^t = \{a^t, b^t\}.$$

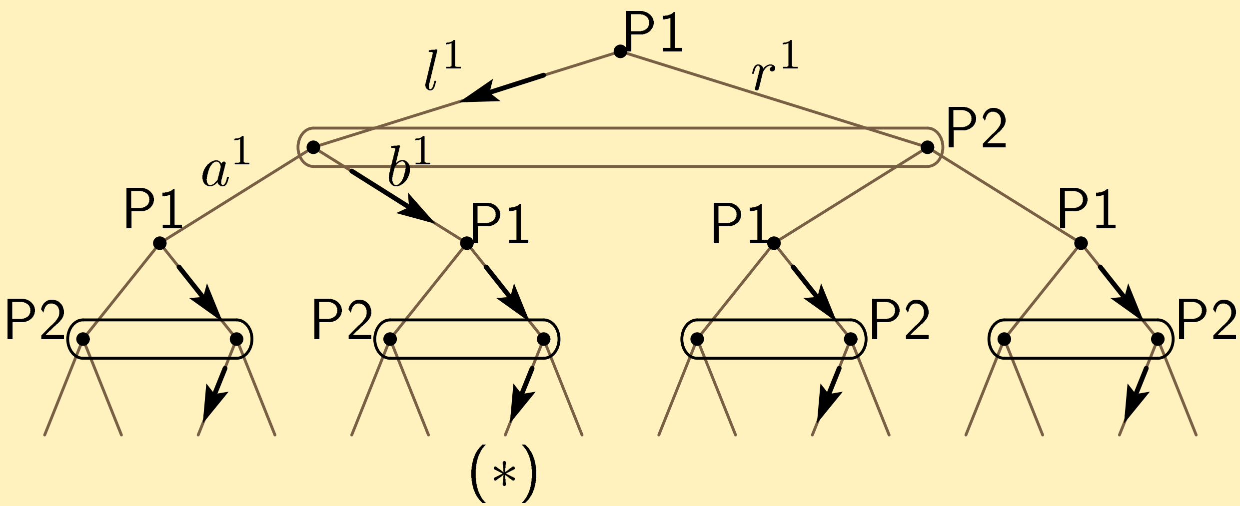


(The overall game)

Multistage games (1)

Illustration of this Proposition $(n = 2,\ t = 1, 2)$

$$\Gamma^t = [N, \{\Delta(S_i^t)\}, \{v_i^t(\cdot)\}],\ S_1^t = \{l^t, r^t\},\ S_2^t = \{a^t, b^t\}.$$



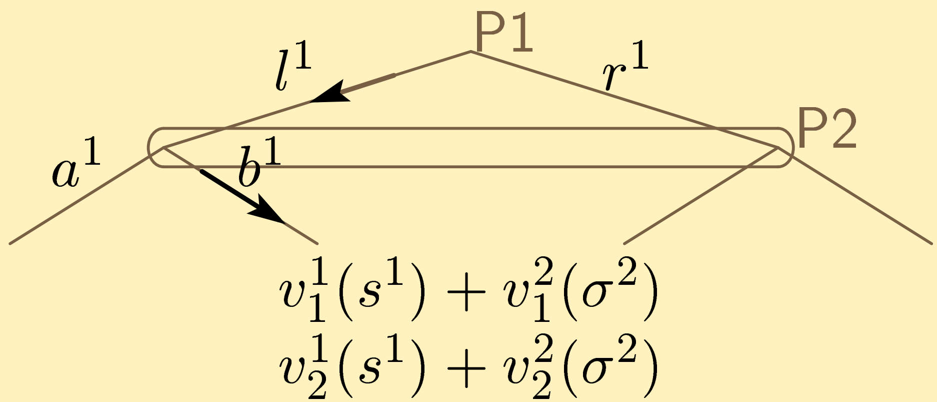
(The overall game)

(l^1, b^1) is a unique NE of Γ_N^1, and (r^2, a^2) is a unique NE of $\Gamma_N^2 \Rightarrow ((l^1; r^2, r^2, r^2, r^2), (b^1; a^2, a^2, a^2, a^2))$ is a unique SPNE.

Multistage games (1)

Illustration of this Proposition $(n = 2,\ t = 1, 2)$

$$\Gamma^t = [N, \{\Delta(S_i^t)\}, \{v_i^t(\cdot)\}],\ S_1^t = \{l^t, r^t\},\ S_2^t = \{a^t, b^t\}.$$

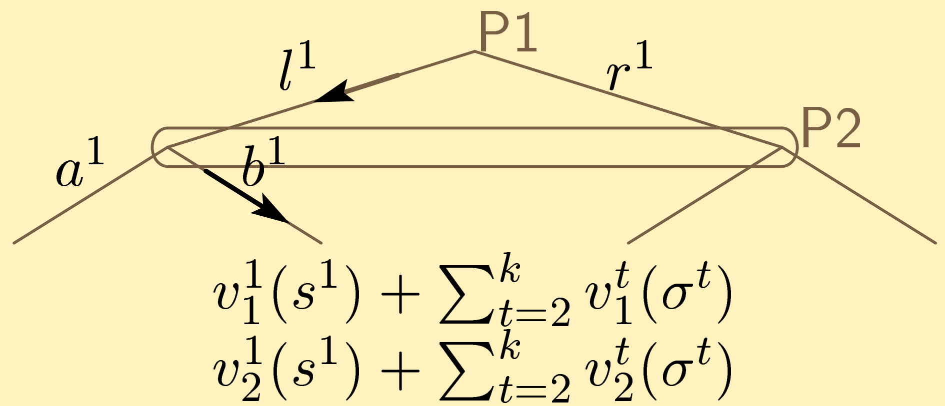


(The reduced game)

(l^1, b^1) is a unique NE of Γ_N^1, and (r^2, a^2) is a unique NE of $\Gamma_N^2 \Rightarrow ((l^1; r^2, r^2, r^2, r^2), (b^1; a^2, a^2, a^2, a^2))$ is a unique SPNE.

Illustration of this Proposition $(n = 2,\ t = 1, 2)$

$$\Gamma^t = [N, \{\Delta(S_i^t)\}, \{v_i^t(\cdot)\}],\ S_1^t = \{l^t, r^t\},\ S_2^t = \{a^t, b^t\}.$$



(Repeat the procedure from $T = 2$ to $T = k$)

(l^1, b^1) is a unique NE of Γ_N^1, and (r^2, a^2) is a unique NE of $\Gamma_N^2 \Rightarrow ((l^1; r^2, r^2, r^2, r^2), (b^1; a^2, a^2, a^2, a^2))$ is a unique SPNE.

Multistage games (1)

Proof of Prop. 9.B.4

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Proof of Prop. 9.B.4 By induction.

- When $T = 1$, it is obvious.

Multistage games (1)

Proof of Prop. 9.B.4 By induction.

- When $T = 1$, it is obvious.
- Assume that our claim is true for all $T \leq k - 1$. We will show that it is true for $T = k$. (we add the $k - 1$ stage game to a simultaneous move game.)

Multistage games (1)

Proof of Prop. 9.B.4 By induction.

- When $T = 1$, it is obvious.
- Assume that our claim is true for all $T \leq k - 1$. We will show that it is true for $T = k$.
- By hypothesis, in any SPNE of the overall game, after play of game Γ^1, the play in the remaining $k - 1$ games must simply involve play of the Nash equilibrium of each game (Prop 9.B.3).

Multistage games (1)

Proof of Prop. 9.B.4 By induction.

- When $T = 1$, it is obvious.
- Assume that our claim is true for all $T \leq k - 1$. We will show that it is true for $T = k$.
- By hypothesis, in any SPNE of the overall game, after play of game Γ^1, the play in the remaining $k - 1$ games must simply involve play of the Nash equilibrium of each game (Prop 9.B.3).
- For each $i \in N$, let G_i be the player i's total payoff in these remaining $k - 1$ games.

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- By hypothesis, in any SPNE of the overall game, after play of game Γ^1, the play in the remaining $k - 1$ games must simply involve play of the Nash equilibrium of each game (Prop 9.B.3).
- In the reduced game that replaces all the subgames that follow Γ^1 with their equilibrium payoffs G_i, player i earns an overall payoff of $u_i(s^1) + G_i$ if s^1 is played at $t = 1$.

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- Player i earns an overall payoff of $u_i(s^1) + G_i$ if s^1 is played at $t = 1$. In this reduced game, the unique NE is σ^{1*}. Therefore, the result holds true for $T = k$.

Multistage games (2)

A two period model $(n = 2,\ t = 1, 2,$ discount factor $\delta)$

Period 1

	m	f
M	$4, 4$	$-1, 5$
F	$5, -1$	$1, 1$

Period 2

	l	g
L	$\mathbf{0}, \mathbf{0}$	$-4, -1$
G	$-1, -4$	$-\mathbf{3}, -\mathbf{3}$

Because the game in the second period contains two Nash equilibria, the equilibrium property of this two period game is quite different from that in the case of Proposition 9.B.4.

Even in this simple two period model, each player has $2 \times 2^4 = 32$ possible strategies, that is, we potentially need to consider $32 \times 32 = 1024$ cases!

Multistage games (2)

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Period 1

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G	$-1, -4$	$-\mathbf{3}, -\mathbf{3}$

Period 2

If δ is not too small, (M, m) and (M, f) can appear in the first stage!

Multistage games (2)

A two period model ($n = 2$, $t = 1, 2$, discount factor δ)

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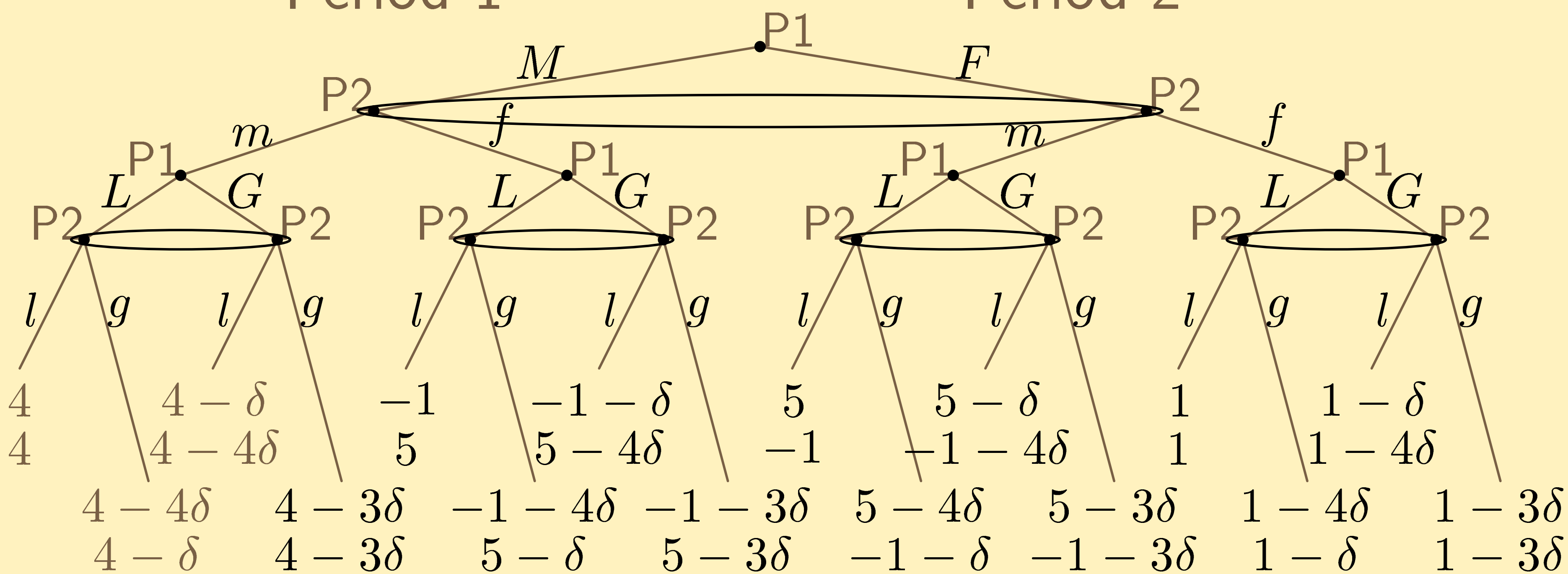
Period 2



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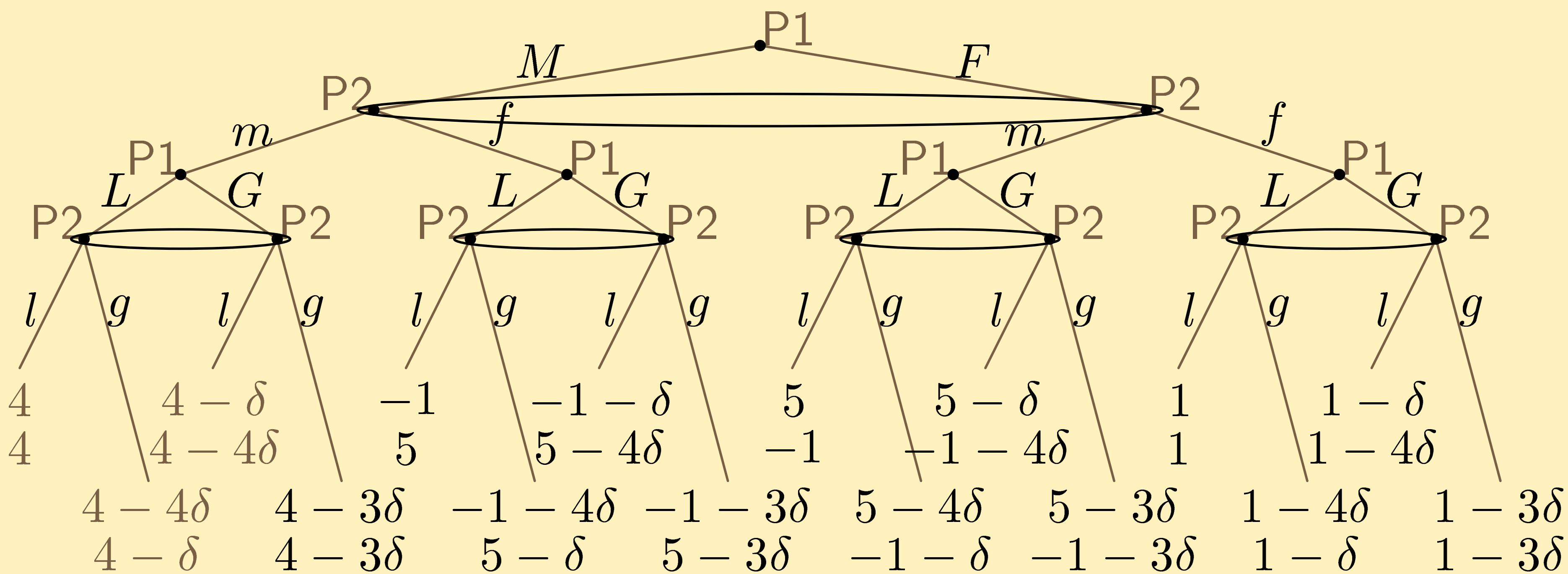
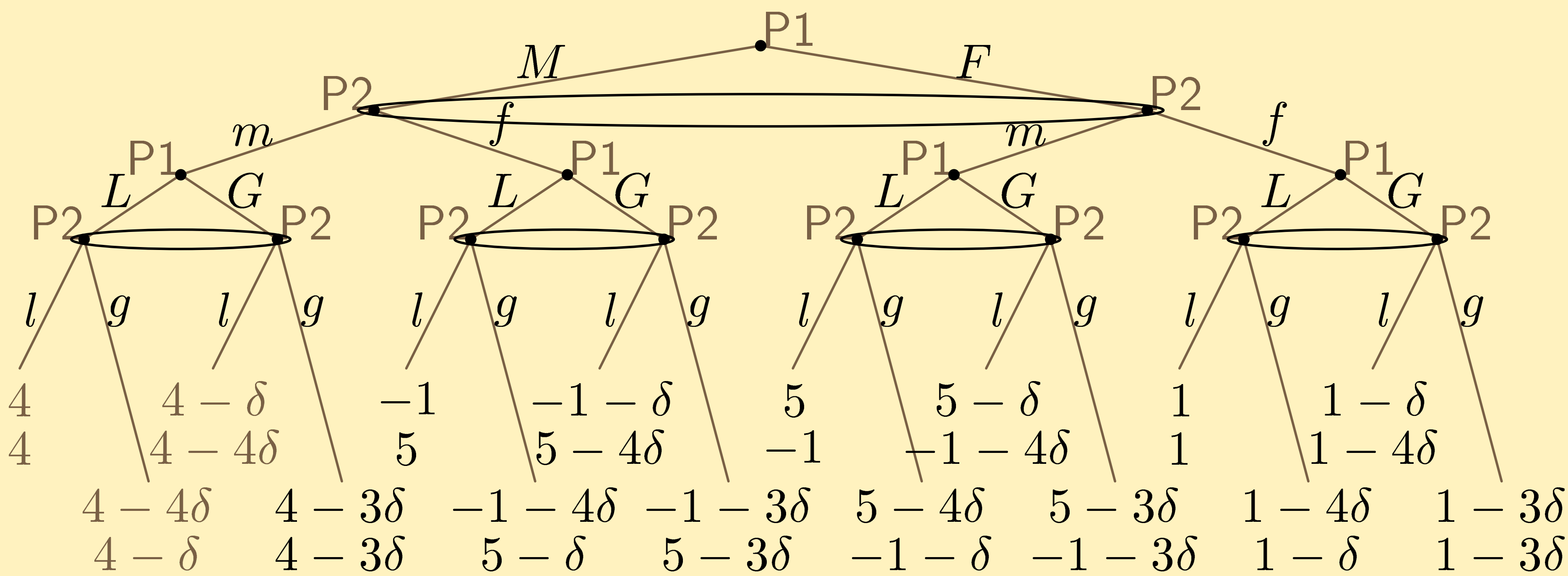


(G, g) in Period 2 is used as a punishment.

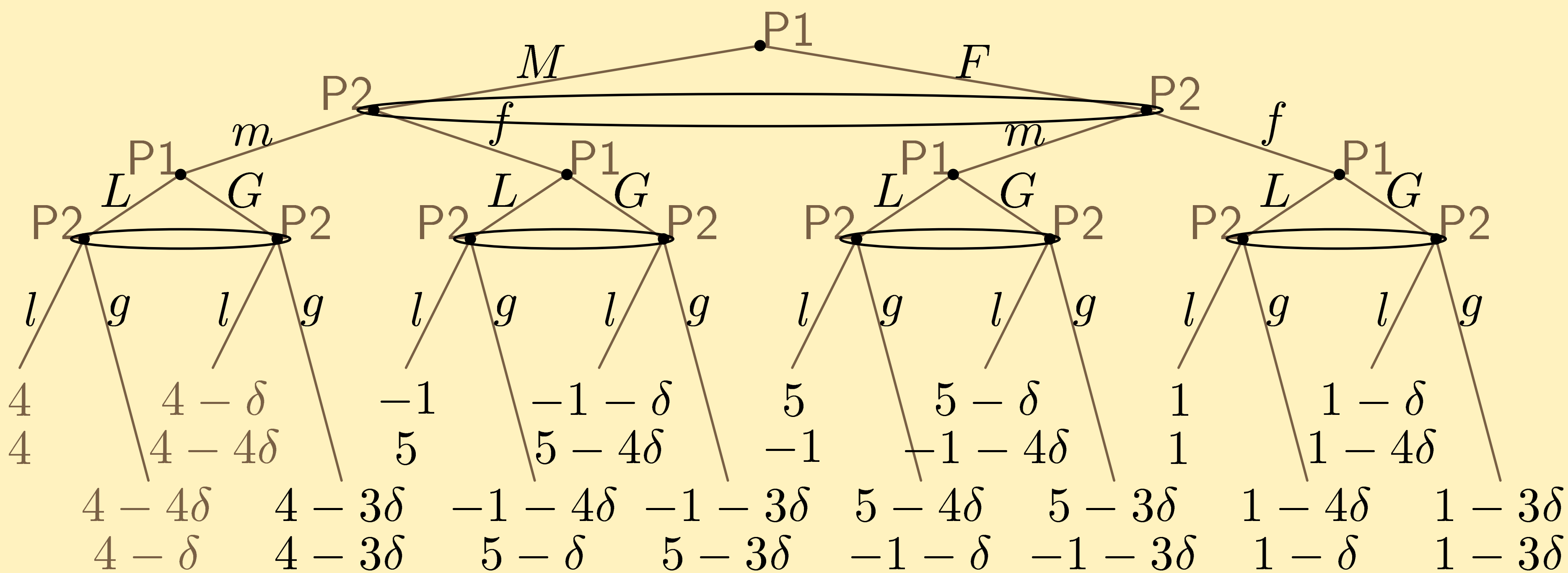
Multistage games (2)



(1) Consider the case in which player 2 sets the following strategy: She plays m in period 1; if (M, m) occurs in period 1, she plays l in period 2, otherwise, she plays g in period 2.

$s_2 = (s_2^1; s_2^2(Mm), s_2^2(Mf), s_2^2(Fm), s_2^2(Ff)) = (m, l, g, g, g)$. What is the best response of player 1?

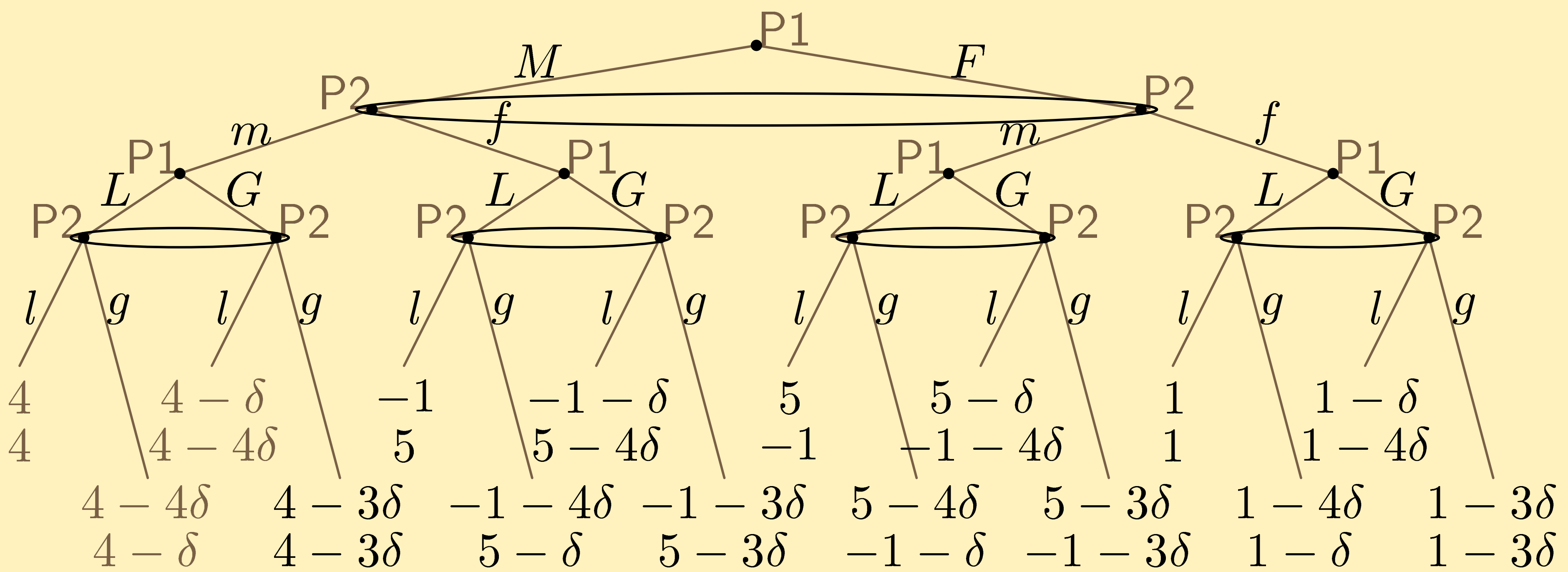
Multistage games (2)



(2) Consider the case in which player 2 sets the following strategy: She plays f in period 1; if (M, f) occurs in period 1, she plays l in period 2, otherwise, she plays g in period 2.

$s_2 = (s_2^1; s_2^2(Mm), s_2^2(Mf), s_2^2(Fm), s_2^2(Ff)) = (f, g, l, g, g)$. What is the best response of player 1?

Multistage games (2)



The following are also SPNE (Props. 9.1-3 on pp.180-1):

(1) $s_1 = (s_1^1, s_1^2(Mm), s_1^2(Mf), s_1^2(Fm), s_1^2(Ff)) = (F, L, L, L, L)$ and $s_2 = (s_2^1, s_2^2(Mm), s_2^2(Mf), s_2^2(Fm), s_2^2(Ff)) = (f, l, l, l, l)$.

(2) $s_1 = (s_1^1, s_1^2(Mm), s_1^2(Mf), s_1^2(Fm), s_1^2(Ff)) = (F, G, G, G, G)$ and $s_2 = (s_1^1, s_1^2(Mm), s_1^2(Mf), s_1^2(Fm), s_1^2(Ff)) = (F, g, g, g, g)$.