Chapter 10: Repeated Games

Outline

- Finitely repeated game
- Infinitely repeated game
- The Fork Theorem
- Examples

Finitely repeated games

Finitely repeated game (def.) Given a stage game G, let $G(T, \delta)$ denote the finitely repeated game in which the stage-game G is played T consecutive times, and δ is the common discount factor.

Example (p.191) A two-stage repeated game. The discount factor is δ.

	M_2	F_2	R_2
M_1	4,4	-1,5	0,0
F_i	5,-1	**1,1**	0,0
R_i	0,0	0,0	**3,3**

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There can be a subgame-perfect outcome of this repeated game in which (M, m) is played in the first stage (if $\delta \geq 1/2$).

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Player i's strategy: Play M_i in stage 1. In stage 2, play R_i if (M_1, M_2) was played in stage 1, and play F_i if anything but (M_1, M_2) was played in stage 1.

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(i) Check the strategy in each 2nd stage game (subgame).

(ii) Check the incentive of player i not to deviate from M_i.

In stage 1, play M_i: $4 + 3\delta$, play F_i: $5 + \delta$.

Infinitely repeated games

Infinitely repeated games The following Prisoners' Dilemma is to be repeated infinitely.

	L_2	R_2
L_1	**1,1**	5,0
R_1	0,5	4,4

δ: the discount factor, which represents the value today of a dollar to be received one period later.

Present value Given the discount factor δ, the present value of the infinite sequence of payoffs $\{v_{i,t}\}_{t=1}^{\infty}$ for player i is

$$v_{i,1} + \delta v_{i,2} + \delta^2 v_{i,3} + \cdots = \sum_{t=1}^{\infty} \delta^{t-1} v_{i,t}.$$

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Average payoff Given the discount factor δ, the average payoff of the infinite sequence of payoffs $\{v_{i,t}\}_{t=1}^{\infty}$ is

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Consider a case in which player i gets y in each period, and the chance continues forever.

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The present value of his payoff is $V = y/(1-\delta)$. The average payoff is y, which is equal to $(1-\delta)V$.

Infinitely repeated games

A (trigger) strategy Play R_i in period 1. In period t, if the outcome of all $t-1$ preceding stages has been (R_1, R_2) then play R_i; otherwise, play L_i.

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Play L_i: yields 5 in this stage; (L_1, L_2) forever.

$$5 + \delta \cdot 1 + \delta^2 \cdot 1 + \cdots = 5 + \delta/(1-\delta).$$

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Play R_i: yields 4 in this stage; the same situation.

$$V = 4 + \delta V \quad \rightarrow \quad V = 4/(1-\delta).$$

Infinitely repeated games

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Infinitely repeated games

Strategy (def.) Consider an infinitely repeated game. Let H_t denote the set of all possible histories of length t, $h_t \in H_t$, and let $H = \cup_{t=1}^{\infty} H_t$ be the set of all possible histories (the union over t of all the sets H_t). A **pure strategy** for player i is a mapping $s_i : H \to S_i$ that maps histories into actions of the stage-game. A **behavioral strategy** of player i, $\sigma_i : H \to \Delta S_i$ maps histories into stochastic choices of actions in each stage.

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Subgame (def.) In a finitely repeated game $G(T, \delta)$, a subgame beginning at stage $t+1$ is the repeated game in which G is played $T-t$ times, denoted $G(T-t, \delta)$.

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There are many subgames that begin at stage $t + 1$, one for each of the possible histories of play through stage t.

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Subgame (def.) In a infinitely repeated game $G(\infty, \delta)$, each subgame beginning at stage $t+1$ is identical to the original game $G(\infty, \delta)$.

There are as many subgames beginning at stage $t+1$ of $G(\infty, \delta)$ as there are possible histories of play through stage t.

Def 10.5 A profile of pure strategies $(s_1^*(\cdot), s_2^*(\cdot), \cdots, s_n^*(\cdot))$, $s_i : H \to S_i$ for all $i \in N$, is a **subgame-perfect equilibrium** if the restriction of $(s_1^*(\cdot), s_2^*(\cdot), \cdots, s_n^*(\cdot))$ is a Nash equilibrium in every subgame.

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Prop. 10.2 Let $G(\infty, \delta)$ be an infinetely repeated game, and let $(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ be a (static) Nash equilibrium strategy profile of the stage-game G. Define the repeated-game strategy for each player i to be the history-independent Nash strategy, $\sigma_i^*(h) = \sigma_i^*$ for all $h \in H$. Then, $(\sigma_1^*(h), \sigma_2^*(h), \ldots, \sigma_n^*(h))$ is a subgame-perfect equilibrium in the repeated game for any $\delta < 1$.

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Prop. 10.3 In an infinitely repeated game $G(\infty, \delta)$, a strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is a subgame-perfect equilibrium if and only if there is no player i and no single history h_{t-1} for which player i would gain from deviating from $s_i(h_{t-1})$.

See the supplemental material related to Section 9.5 (one-stage deviation principle).

Infinitely repeated games

Trigger-strategy We must show that the trigger strategies constitute a Nash equilibrium on every subgame of that infinitely repeated game.

	L_2	R_2
L_1	**1,1**	5,0
R_1	0,5	4,4

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Trigger-strategy Play R_i in period 1. In period t, if the outcome of all $t-1$ preceding stages has been (R_1, R_2) then play R_i; otherwise, play L_i.

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1. subgames in which all the outcomes of earlier stages have been (R_1, R_2).
2. subgames in which the outcome of at least one earlier stage differs from (R_1, R_2).

The Folk Theorem

Def. 10.6 Consider two vectors $v = (v_1, v_2, \ldots, v_n)$ and $v' = (v'_1, v'_2, \ldots, v'_n)$ in $\Re^n$. The vector $\hat{v} = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$ is a **convex conbination** of v and v' if there exists some number $\alpha \in [0, 1]$ such that

$$\hat{v} = \alpha v + (1 - \alpha) v',$$

$$or \quad \hat{v}_i = \alpha v_i + (1 - \alpha) v'_i \ \text{ for all } i \in \{1, 2, \cdots, n\}.$$

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Def. 10.7 Given a set of vectors $V = \{v^1, v^2, \cdots, v^k\}$ in $\Re^n$, the **convext hull** of V is the smallest convex set that contains all the vectors in V.

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$$CoHull(V)$$

$$= \{v \in \Re^n : \exists(\alpha_1, \cdots, \alpha_k) \in \Re^k, \sum_{j=1}^{k} \alpha_j = 1, \text{ such that } v = \sum_{j=1}^{k} \alpha_j v^j\}.$$

The Folk Theorem

Feasible The payoffs $(x_1, \ldots, x_N)$ are feasible in the stage game G if they are a convex combination of the pure-strategy payoffs of G.

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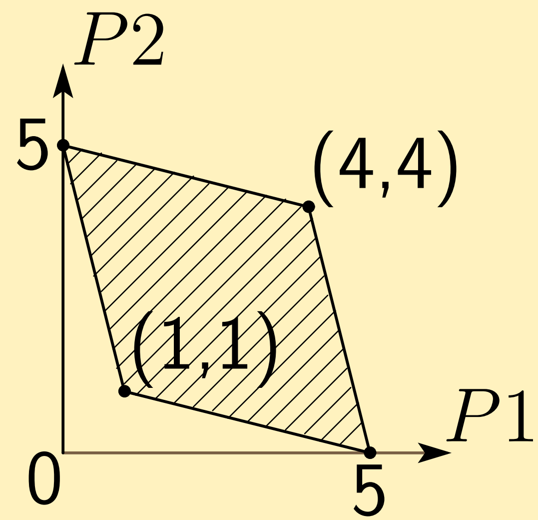

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Theorem 10.1 (Friedman 1971) Let G be a finite, static game of complete information. Let $(e_1, \ldots, e_n)$ denote the payoffs from a Nash equilibrium of G, and let $(x_1, \ldots, x_n)$ denote any other feasible payoffs from G.

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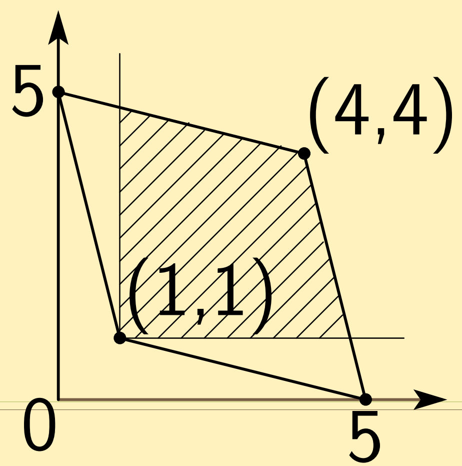

Example

Collusion $Q = q_1 + q_2$, $P = a - Q$, $MC = c$.

NE: $q_C = (a - c)/3$ Monopoly: $q_m = (a - c)/2$.

NE: $\pi_C = (a - c)^2/9$ Collusion: $\pi_m/2 = (a - c)^2/8$.

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Trigger strategy In period 1, produce $q_m/2$. In period t, produce $q_m/2$ if they have produced $q_m/2$ in each of the $t-1$ previous periods; otherwise, produce q_C.

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Deviation Given the rival produces $q_m/2$, the optimal quantity to deviate is

$$\arg\max_{q_j} \left(a - q_j - \frac{q_m}{2} - c\right) q_j \quad \rightarrow \quad q_j = \frac{3(a-c)}{8}.$$

The profit of the deviating firm is $\pi_d = 9(a-c)^2/64$.

Example (cont.)

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Cooperation The firms play the trigger strategy if

$$\frac{1}{1-\delta}\frac{\pi_m}{2} \geq \pi_d + \frac{\delta}{1-\delta}\pi_C \quad \rightarrow \quad \delta \geq \frac{9}{17}.$$

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If $\delta < 9/17$: The following trigger strategy is useful.

In period 1, produce q^*. In period t, produce q^* if they have produced q^* in each of the $t-1$ previous periods; otherwise, produce q_C. $(q_m/2 < q^* < q_C)$