Chapter 12: Bayesian Games

Outline

- Static Bayesian Game of incomplete information. Type, prior, Bayesian Nash equilibrium, conditional probability.
- Example
- Mixed strategy revisited
- Adverse Selection (12.3 Tedelis and 13.B Mas-Colell et al.).

Static games of incomplete information

Gibbons Chap. 3.1 Asymmetric Information Cournot.

$P(Q) = a - Q$, where $Q = q_1 + q_2$. $C_1(q_1) = cq_1$.

$$C_2(q_2) = \begin{cases} c_H q_2 & \text{with probability } k, \\ c_L q_2 & \text{with probability } 1-k, \end{cases} \qquad (c_L < c_H),$$

$a > 0$, $c \in (0, a/2)$, $c_L \in (0, a/2)$, $c_H \in (0, a/2)$.

Static games of incomplete information

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$$C_2(q_2) = \begin{cases} c_H q_2 & \text{with probability } k, \\ c_L q_2 & \text{with probability } 1-k, \end{cases} \qquad (c_L < c_H),$$

Firm 2 knows c_j although firm 1 does only the probability distribution of c_j $(j = H, L)$.

Static games of incomplete information

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$$q_{2H}^* = \arg\max_{q_2} (a - q_1^* - q_2 - c_H)q_2,$$
$$q_{2L}^* = \arg\max_{q_2} (a - q_1^* - q_2 - c_L)q_2.$$

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$$q_{2H}^* = \arg\max_{q_2} \, (a - q_1^* - q_2 - c_H) q_2,$$

$$q_{2L}^* = \arg\max_{q_2} \, (a - q_1^* - q_2 - c_L) q_2.$$

$$q_1^* = \arg\max_{q_1} \; k(a - q_1 - q_{2H}^* - c) q_1 + (1-k)(a - q_1 - q_{2L}^* - c) q_1.$$

Static games of incomplete information

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$$q_1^* = \arg\max_{q_1} \; k(a - q_1 - q_{2H}^* - c) q_1 + (1-k)(a - q_1 - q_{2L}^* - c) q_1.$$

BR: $q_{2H}^* = (a - q_1^* - c_H)/2, \ldots$

Static games of incomplete information

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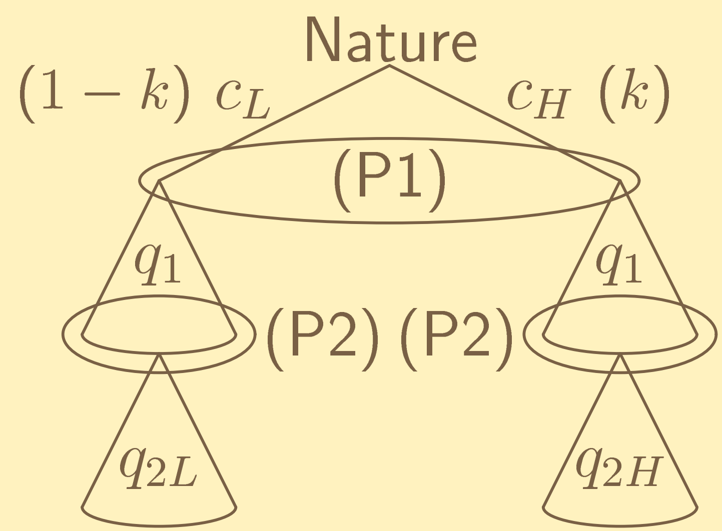

Important assumption

The following is just a summary of the content on pp. 241-6.

- Despite each player not necessary knowing the actual preferences of his opponents, he knows the precise way in which Nature chooses these preferences.
  Each player knows the probability distribution over actual preferences (types).
  Common prior assumption: all the players agree on the way the world works, as described by the probabilities according to which Nature chooses the different types of players.

Important assumption

The following is just a summary of the content on pp. 241-6.

- Despite each player not necessary knowing the actual preferences of his opponents, he knows the precise way in which Nature chooses these preferences.
- We cannot perform equilibrium analysis unless we assume that each player knows the distribution of his opponents' types (the common prior assumption).

  By imposing this assumption, Harsányi changed the complex and challenging concept of incomplete information into a well-known game of *imperfect information*.

Elements of static Bayesian games

Static Bayesian Games

$$[N, \{A_i\}_{i=i}^{n}, \{\Theta_i\}_{i=1}^{n}, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^{n}, \{\phi_i\}_{i=1}^{n}]$$

Elements of static Bayesian games

Static Bayesian Games

$$[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$$

$v_i(a_1 \ldots, a_n; \theta_i)$: player i's possible payoff functions.

Elements of static Bayesian games

Static Bayesian Games

$[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$

$v_i(a_1 \ldots, a_n; \theta_i)$: player i's possible payoff functions.

θ_i: player i's **type** $(\theta_i \in \Theta_i)$. (ex. $\Theta_2 = \{c_H, c_L\})$

θ_i is privately known by player i

Elements of static Bayesian games

Static Bayesian Games

$[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$

$v_i(a_1 \ldots, a_n; \theta_i)$: player i's possible payoff functions.

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$\theta_{-i} \in \Theta_{-i}$: the types of the other players

Elements of static Bayesian games

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$[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$

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θ_i is privately known by player i

$\theta_{-i} \in \Theta_{-i}$: the types of the other players

$\phi_i(\theta_{-i}|\theta_i)$: player i's **belief** (the prob. distribution).

i's uncertainty about the other players' possible types, θ_{-i}, given i's own type θ_i.

(ex. $\phi_1(\theta_2 = c_H) = k$ and $\phi_1(\theta_2 = c_L) = 1 - k$).

Elements of static Bayesian games

Static Bayesian Games

$[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$

$v_i(a_1 \ldots, a_n; \theta_i)$: player i's possible payoff functions.

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$\theta_{-i} \in \Theta_{-i}$: the types of the other players

$\phi_i(\theta_{-i}|\theta_i)$: player i's **belief** (the prob. distribution).

i's uncertainty about the other players' possible types, θ_{-i}, given i's own type θ_i.

Firm 1 directly uses the prob. distribution of c_j because it cannot update its belief.

Belief updates

The timing of a static Bayesian game

1. Nature draws a type vector $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_i \in \Theta_i$.
2. Nature reveals θ_i to player i but not to any other.
3. Player i simultaneously chooses $a_i \in A_i$.

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2. Nature reveals θ_i to player i but not to any other.
3. Player i simultaneously chooses $a_i \in A_i$.

Remarks

1. There are games in which player i has private information not only his/her **own** payoff function but also about **another** player's payoff function.
   Ex.) $p = a_i - Q$ $(i = H, L)$, only firm 2 knows a_i.

Belief updates

The timing of a static Bayesian game

1. Nature draws a type vector $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i \in \Theta_i$.
2. Nature reveals θ_i to player i but not to any other.
3. Player i simultaneously chooses $a_i \in A_i$.

Remarks

1. There are games in which player i has private information not only his/her **own** payoff function but also about **another** player's payoff function.
2. Player i computes/updates the belief using Bayes's rule (compute the conditional probability):

$$\phi_i(\theta_{-i}|\theta_i) = \frac{\phi(\theta_{-i},\theta_i)}{\phi(\theta_i)} = \frac{\phi(\theta_{-i},\theta_i)}{\sum_{\theta_{-i}\in\Theta_{-i}} \phi(\theta_{-i},\theta_i)}.$$

Bayes's rule

Player i computes the belief $\phi_i(\theta_{-i}|\theta_i)$ using Bayes's rule:

$$\phi_i(\theta_{-i}|\theta_i) = \frac{\phi(\theta_{-i}, \theta_i)}{\phi(\theta_i)} = \frac{\phi(\theta_{-i}, \theta_i)}{\sum_{\theta_{-i} \in \Theta_{-i}} \phi(\theta_{-i}, \theta_i)}.$$

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Example $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$.

$\phi(\cdot)$	L	R
U	0.4	0.3
D	0.2	0.1

$$\phi(U) = 0.4 + 0.3 = 0.7$$
$$\phi(D) = 0.2 + 0.1 = 0.3$$

$$\phi_1(L|U) = \frac{\phi(L,U)}{\phi(U)} = \frac{0.4}{0.7} = \frac{4}{7},\ \phi_1(R|D) = \frac{\phi(R,D)}{\phi(D)} = \frac{1}{3}.$$

Strategy in static Bayesian games

Strategy In the static Bayesian game $[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$, a **strategy** for player i is a function $s_i(\theta_i)$, where for each type $\theta_i \in \Theta_i$, $s_i(\theta_i)$ specifies the action from the feasible set A_i that type θ_i would choose if drawn by nature.

Strategy in static Bayesian games

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Each player chooses his type-contingent strategy before he learns his type.

Strategy in static Bayesian games

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S_i: The set of all possible functions with domain Θ_i and range A_i.

Cournot ex.: Player 2's (equilibrium) strategy (q_{2H}^*, q_{2L}^*)

Strategy in static Bayesian games

Strategy In the static Bayesian game $[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$, a **strategy** for player i is a function $s_i(\theta_i)$, where for each type $\theta_i \in \Theta_i$, $s_i(\theta_i)$ specifies the action from the feasible set A_i that type θ_i would choose if drawn by nature.

S_i: The set of all possible functions with domain Θ_i and range A_i.

A separating strategy Each type $\theta_i \in \Theta_i$ chooses a different action $a_i \in A_i$.

A pooling strategy All types choose the same action.

Bayesian Nash equilibrium

Bayesian Nash eq. In the static Bayesian game $[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$, the strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a **Bayesian Nash equilibrium** if for each player i and for each of i's types θ_i in Θ_i, $s_i^*(\theta_i)$ solves

$$\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} v_i(s_i^*(\theta_1), \ldots, s_{i-1}^*(\theta_{i-1}), a_i, s_{i+1}^*(\theta_{i+1}), \ldots, s_n^*(\theta_n); \theta_i)\phi_i(\theta_{-i}|\theta_i).$$

Bayesian Nash equilibrium

Bayesian Nash eq. In the static Bayesian game $[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$, the strategies $s^* = (s_1^*, \ldots, s_n^*)$ are a **Bayesian Nash equilibrium** if for each player i and for each of i's types θ_i in Θ_i, $s_i^*(\theta_i)$ solves

$$\max_{a_i \in A_i} \sum_{\theta_{-i} \in \Theta_{-i}} v_i(s_i^*(\theta_1), \ldots, s_{i-1}^*(\theta_{i-1}), a_i, s_{i+1}^*(\theta_{i+1}), \ldots, s_n^*(\theta_n); \theta_i)\phi_i(\theta_{-i}|\theta_i).$$

That is, no player wants to change his/her strategy, even if the change involves only one action by one type.

$$s_i = (s_i(\theta_i^1), \ldots, s_i(\theta_i^{\#\theta_i})).$$

Expected payoff

Expected payoff The expected payoff when a strategy profile $s(\cdot) = (s_1(\cdot), \dots, s_n(\cdot))$ is played is

$$\begin{aligned}
\tilde{u}_i(s(\cdot)) &= E_{\theta=(\theta_i,\theta_{-i})}[v_i(s(\theta);\theta)] \\
&= \sum_{\theta\in\Theta} \phi(\theta) v_i(s(\theta);\theta) \\
&= \sum_{\theta_i\in\Theta_i} \sum_{\theta_{-i}\in\Theta_{-i}} \phi(\theta_i,\theta_{-i}) v_i(s(\theta);\theta).
\end{aligned}$$

Converting into complete information games

Expected payoff

$$\tilde{v}_i(s(\cdot)) = \sum_{\theta_i \in \Theta_i} \sum_{\theta_{-i} \in \Theta_{-i}} \phi(\theta_i, \theta_{-i}) v_i(s(\theta); \theta).$$

Bayesian Nash eq. In the static Bayesian game $[N, \{A_i\}_{i=i}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot; \theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n]$, a Bayesian Nash equilibrium is a strategy profile $s(\cdot) = (s_1(\cdot), \ldots, s_n(\cdot))$ that constitute a Nash equilibrium of the complete information game, $[N, \{S_i\}_{i \in N}, \{\tilde{v}_i\}_{i \in N}]$, that is, $\forall i \in N$ and all $\hat{s}_i(\cdot) \in S_i$,

$$\tilde{v}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{v}_i(\hat{s}_i(\cdot), s_{-i}(\cdot)).$$

$$s_i = (s_i(\theta_i^1), \ldots, s_i(\theta_i^{\#\theta_i})).$$

Example

Chicken game (complete information) (C, d) and (D, c) are pure strategy Nash equilibria $(R > 0)$.

1/2	c	d
C	$0, 0$	$0, R$
D	$R, 0$	$-1, -1$

Example

Chicken game (incomplete information) The payoffs in which (D, d) occurs are private information.

$1/2$	c	d
C	$0, 0$	$0, R$
D	$R, 0$	x_1, x_2

Example

Chicken game (incomplete information) Player i has two types: H and L. Each of them is assigned with probability 1/2. When his type is H (resp. L), his payoff in which (D, d) occurs is $R/2 - h$ (resp. $R/2 - l$) $(h > l > 0)$.

$(\theta_1, \theta_2) = (L, L)$

1/2	c	d
C	$0, 0$	$0, R$
D	$R, 0$	$\frac{R}{2} - l, \frac{R}{2} - l$

$(\theta_1, \theta_2) = (L, H)$

1/2	c	d
C	$0, 0$	$0, R$
D	$R, 0$	$\frac{R}{2} - l, \frac{R}{2} - h$

$(\theta_1, \theta_2) = (H, L)$

1/2	c	d
C	$0, 0$	$0, R$
D	$R, 0$	$\frac{R}{2} - h, \frac{R}{2} - l$

$(\theta_1, \theta_2) = (H, H)$

1/2	c	d
C	$0, 0$	$0, R$
D	$R, 0$	$\frac{R}{2} - h, \frac{R}{2} - h$

Example

Chicken game (incomplete information) When his type is H (resp. L), his payoff in which (D, d) occurs is $R/2 - h$ (resp. $R/2 - l$) ($h > l > 0$).

Player 1's strategy: $x_1 y_1 \in S_1 = \{CC, CD, DC, DD\}$, where x_1 (resp. y_1) is the action when his type is L (resp. H).

Player 2's strategy: $x_2 y_2 \in S_2 = \{cc, cd, dc, dd\}$, where x_2 (resp. y_2) is the action when his type is L (resp. H).

Example

Chicken game (incomplete information) When his type is H (resp. L), his payoff in which (D, d) occurs is $R/2 - h$ (resp. $R/2 - l$) $(h > l > 0)$.
Player 2's strategy: $x_2 y_2 \in S_2 = \{cc, cd, dc, dd\}$, where x_2 (resp. y_2) is the action when his type is L (resp. H).

For instance, the expected payoffs in which player 1 chooses DC and player 2 chooses dd are

$$\begin{aligned} Ev_1(DC, dd) &= \frac{1}{4}v_1(D, d; L) + \frac{1}{4}v_1(D, d; L) \\ &\quad + \frac{1}{4}v_1(C, d; H) + \frac{1}{4}v_1(C, d; H) \\ &= (R - 2l)/4. \end{aligned}$$

Example

Chicken game (incomplete information) When his type is H (resp. L), his payoff in which (D, d) occurs is $R/2 - h$ (resp. $R/2 - l$) ($h > l > 0$).
Player 2's strategy: $x_2 y_2 \in S_2 = \{cc, cd, dc, dd\}$, where x_2 (resp. y_2) is the action when his type is L (resp. H).

1/2	cc	cd	dc	dd
CC	$0, 0$	$0, \frac{R}{2}$	$0, \frac{R}{2}$	$0, R$
CD	$\frac{R}{2}, 0$	$\frac{3R-2h}{8}, \frac{3R-2h}{8}$	$\frac{3R-2h}{8}, \frac{3R-2l}{8}$	$\frac{R-2h}{4}, \frac{3R-l-h}{4}$
DC	$\frac{R}{2}, 0$	$\frac{3R-2l}{8}, \frac{3R-2h}{8}$	$\frac{3R-2l}{8}, \frac{3R-2l}{8}$	$\frac{R-2l}{4}, \frac{3R-l-h}{4}$
DD	$R, 0$	$\frac{3R-l-h}{4}, \frac{R-2h}{4}$	$\frac{3R-l-h}{4}, \frac{R-2l}{4}$	$\frac{R-l-h}{2}, \frac{R-l-h}{2}$

Mixed Strategies Revisited

Matching Pennies The unique mixed strategy Nash equilibrium has each player plays H with prob. $1/2$.

$1/2$	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Mixed Strategies Revisited

Matching Pennies The unique mixed strategy Nash equilibrium has each player plays H with prob. $1/2$.

$1/2$	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Players are indifferent between H and T.
Why do they choose H and T randomly even when those are indifferent for them?

Mixed Strategies Revisited

Matching Pennies The unique mixed strategy Nash equilibrium has each player plays H with prob. $1/2$.

Harsányi's interpretation The original game with perfectly known payoffs should be considered an idealization, a limit, of nearby games with a small amount of payoff uncertainty.

In the context of Matching Pennies, each player may have some slight preference for choosing H over T or choosing T over H.

Mixed Strategies Revisited

Harsányi's interpretation Both ε_1 and ε_2 are independent and uniformly distributed on $[-\varepsilon, \varepsilon]$ for some small $\varepsilon > 0$. ε_i is privately known to player i.

$1/2$	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

Mixed Strategies Revisited

Harsányi's interpretation

1/2	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

Consider pure strategy of player i (s_i: $\Theta_i \to A_i$),
where $\Theta_i = [-\varepsilon, \varepsilon]$, $A_i = \{H, T\}$.

As a result, player i chooses H with some probability.

Mixed Strategies Revisited

Harsányi's interpretation

$1/2$	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

The expected payoffs of player 1 with ε_1 (q is the prob. of playing H by player 2; q depends on s_2)

$$\begin{aligned} v_1(H, s_2; \varepsilon_1) &= q(1+\varepsilon_1) + (1-q)(-1+\varepsilon_1) \\ &= (2q-1) + \varepsilon_1, \\ v_1(L, s_2; \varepsilon_1) &= q(-1) + (1-q) = (1-2q). \end{aligned}$$

H is better for player 1 iff $\varepsilon_1 > 2(1-2q)$.

Mixed Strategies Revisited

Harsányi's interpretation

1/2	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

H is better for player 1 iff $\varepsilon_1 > 2(1-2q)$, where q is the prob. of playing H by player 2.
H is better for player 2 iff $\varepsilon_2 > 2(1-2p)$, where p is the prob. of playing H by player 1.

Consider the pure strategy: Player i plays H iff ε_i is larger than $\bar{\varepsilon_i}$. $p = (\varepsilon - \bar{\varepsilon_1})/(2\varepsilon)$, $q = (\varepsilon - \bar{\varepsilon_2})/(2\varepsilon)$, $2(1-2q) = \bar{\varepsilon_1}$, $2(1-2p) = \bar{\varepsilon_2}$.

Mixed Strategies Revisited

Harsányi's interpretation

1/2	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

H is better for player 1 iff $\varepsilon_1 > 2(1-2q)$; H is better for player 2 iff $\varepsilon_2 > 2(1-2p)$.

Consider the pure strategy: Player i plays H iff ε_i is larger than $\bar{\varepsilon_i}$.

$p = (\varepsilon - \bar{\varepsilon_1})/(2\varepsilon)$, $q = (\varepsilon - \bar{\varepsilon_2})/(2\varepsilon)$, $2(1-2q) = \bar{\varepsilon_1}$, $2(1-2p) = \bar{\varepsilon_2}$. $\Rightarrow$ $\varepsilon_1 = \varepsilon_2 = 0$.

Mixed Strategies Revisited

Harsányi's interpretation

1/2	H	T
H	$1+\varepsilon_1, -1+\varepsilon_2$	$-1+\varepsilon_1, 1$
T	$-1, 1+\varepsilon_2$	$1, -1$

Claim 12.3 In the Bayesian perturbed Matching Pennies game, there is a unique pure-strategy Bayesian Nash equilibrium in which $s_i(\varepsilon_i) = H$ if and only if $\varepsilon_i \geq 0$ and $s_i(\varepsilon_i) = L$ if and only if $\varepsilon_i < 0$.
This equilibrium converges in outcomes and payoffs to the Matching Pennies game when $\varepsilon \to 0$.

Inefficient Trade and Adverse Selection

Trade problem with incomplete information Player 1 has a car. Its quality is privately known to him. The quality may be low (L), mediocre (M), or high (H), each with equal probability $1/3$. The type set of player 1 is $\Theta_1 = \{L, M, H\}$. Player 2 is the buyer of the car.

The monetary values for the car are given as

$$v_1(\theta_1) = \begin{cases} 10 & \text{if } \theta_1 = L, \\ 20 & \text{if } \theta_1 = M, \\ 30 & \text{if } \theta_1 = H, \end{cases} \quad v_2(\theta_1) = \begin{cases} 14 & \text{if } \theta_1 = L, \\ 24 & \text{if } \theta_1 = M, \\ 34 & \text{if } \theta_1 = H. \end{cases}$$

If the quality is common knowledge, trade is beneficial.

Inefficient Trade and Adverse Selection

Trade problem with incomplete information

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1. Nature sets the quality, then player 1 knows it.
2. Player 2 offers price p to player 1.
3. Player 1 decides whether to accept the price (A) or reject it (R)

Inefficient Trade and Adverse Selection

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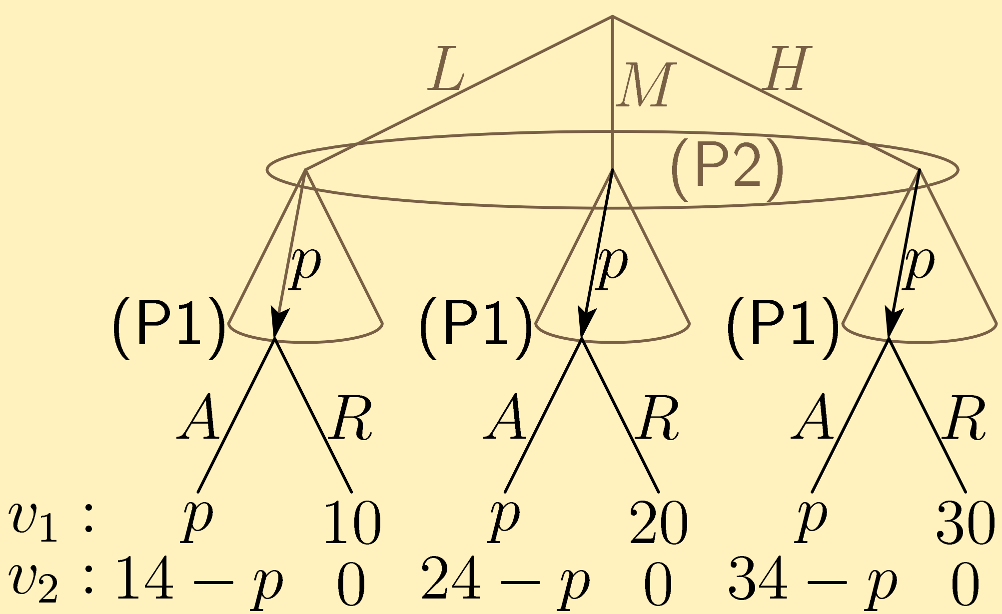

Player 1's strategy space

$S_1 = \{A, R\}$.

$s_1 : [0, \infty) \times \Theta_1 \to \{A, R\}$.

Player 2's strategy space

$S_2 = [0, \infty)$.

Inefficient Trade and Adverse Selection

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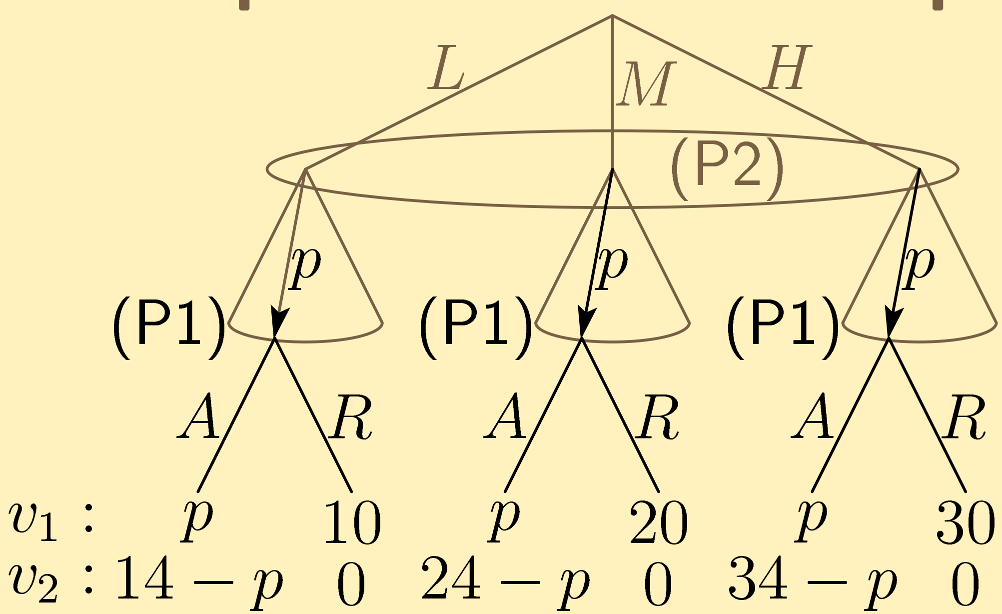

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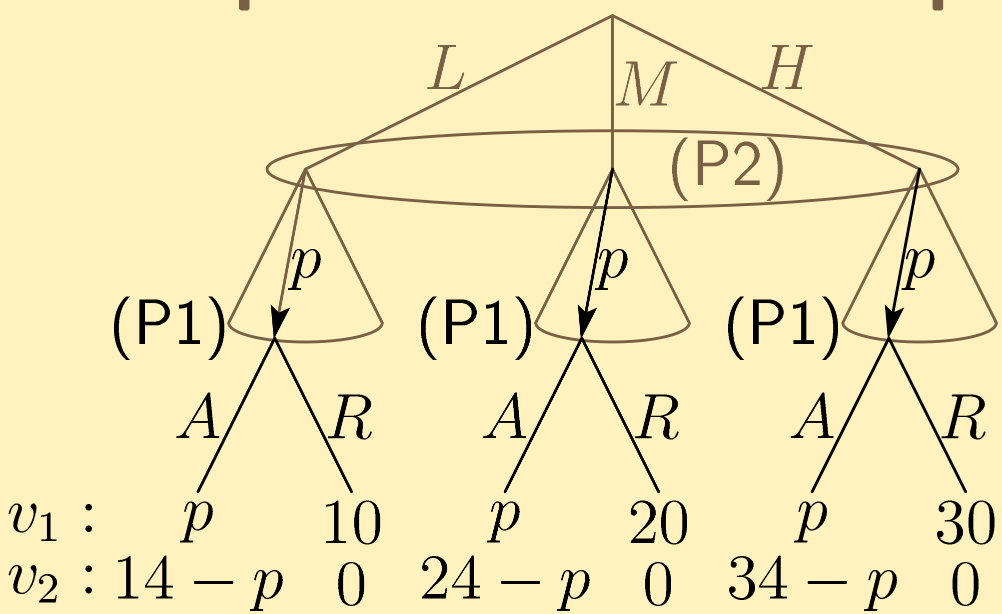
Consider the following strategy of player 1 (the bold font differs from $s_1(\theta_1)$ on p.260).

$$s_1(\theta_1) = \begin{cases} A \text{ if } p \geq \mathbf{10}\text{, otherwise, } R & \text{when } \theta_1 = L, \\ A \text{ if } p \geq 20\text{, otherwise, } R & \text{when } \theta_1 = M, \\ A \text{ if } p \geq 30\text{, otherwise, } R & \text{when } \theta_1 = H. \end{cases}$$

Inefficient Trade and Adverse Selection

Trade problem with incomplete information



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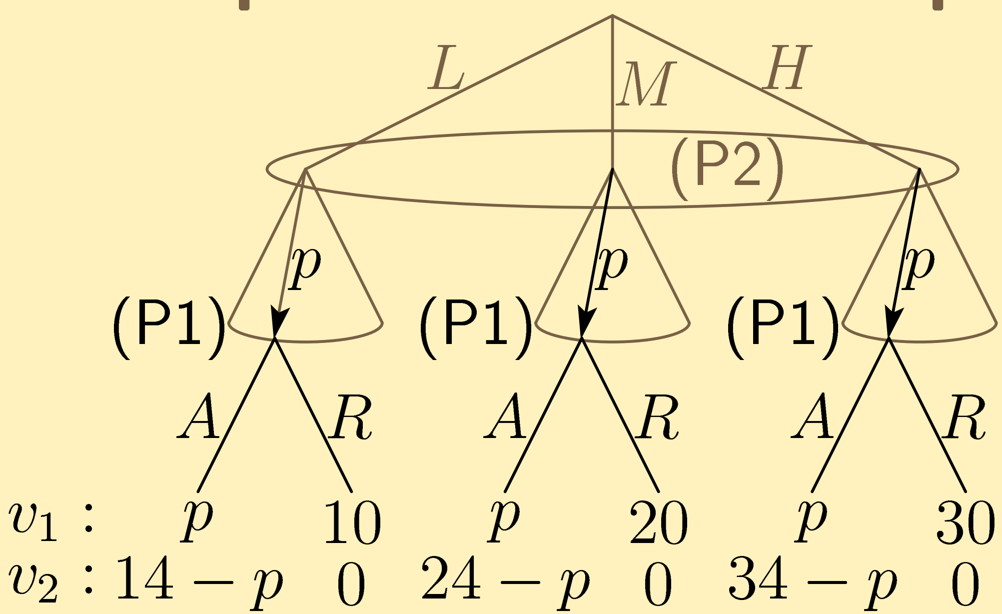
Note that $s_1(\theta_1)$ on p.260 is correct.

Trade problem with incomplete information



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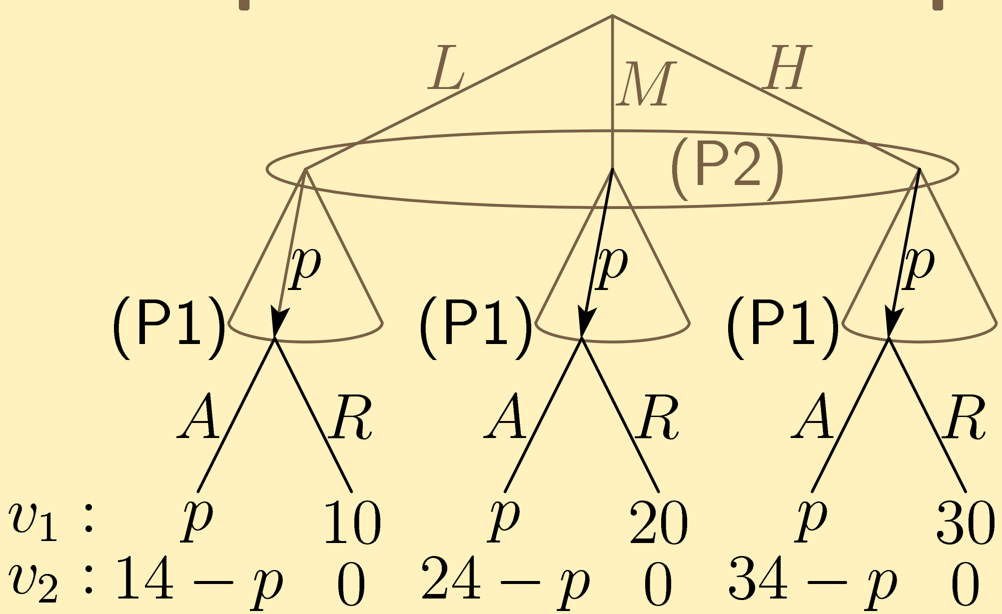
The set of "reasonable" prices for player 2 is $\{10, 20, 30\}$.

Inefficient Trade and Adverse Selection

Trade problem with incomplete information



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$$
\begin{array}{llll}
p = 10: & \Pr(A) = 1/3, & E[v_2(\theta_1)] = 14, & (14 - 10)/3 = 4/3; \\
p = 20: & \Pr(A) = 2/3, & E[v_2(\theta_1)] = 19, & 2(19 - 20)/3 = -2/3; \\
p = 30: & \Pr(A) = 1, & E[v_2(\theta_1)] = 24, & 24 - 30 = -6.
\end{array}
$$

$p = 10$ is the best for player 2 if player 1 employs $s_1(\theta_1)$ mentioned above.

Adverse Selection (MWG 13.B)

MWG §13.B Informational asymmetries and adverse selection

Example: Labor market

- Many identical firms.
- Many workers with different productivities.

N:	Total number of workers,
$\Theta = [\underline{\theta}, \bar{\theta}]$:	Set of workers' types ($0 \leq \underline{\theta} < \bar{\theta} < \infty$).

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A hired type $\theta \in \Theta$ worker produces θ units.

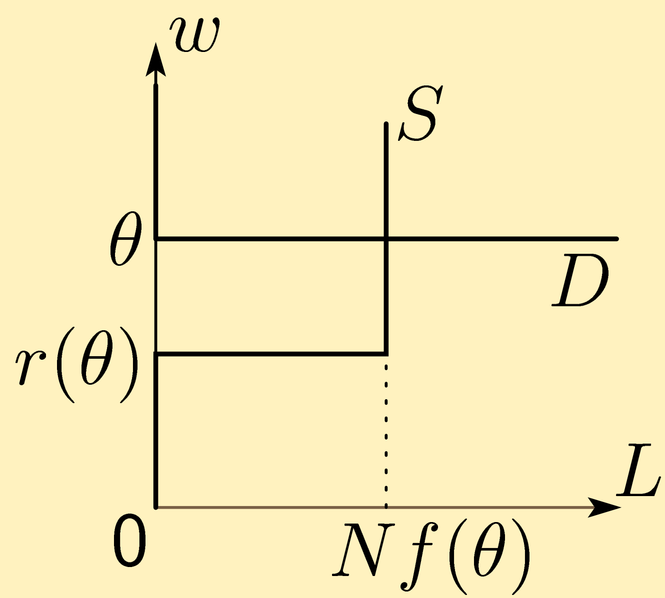
$r(\theta)$:	The reservation wage of type θ,
$w(\theta)$:	The wage type θ receives,
$\theta - w(\theta)$:	The profit of a firm hiring a type θ worker,
$F(\theta)$:	The distribution function on Θ, $f(\theta) = F'(\theta)$.

Complete information (benchmark)

Type θ A market for type θ exists.

The equilibrium wage: $w(\theta) = \theta \ \forall \theta \in \Theta$.

The set of workers: $\{\theta \in \Theta | r(\theta) \leq \theta\}$.



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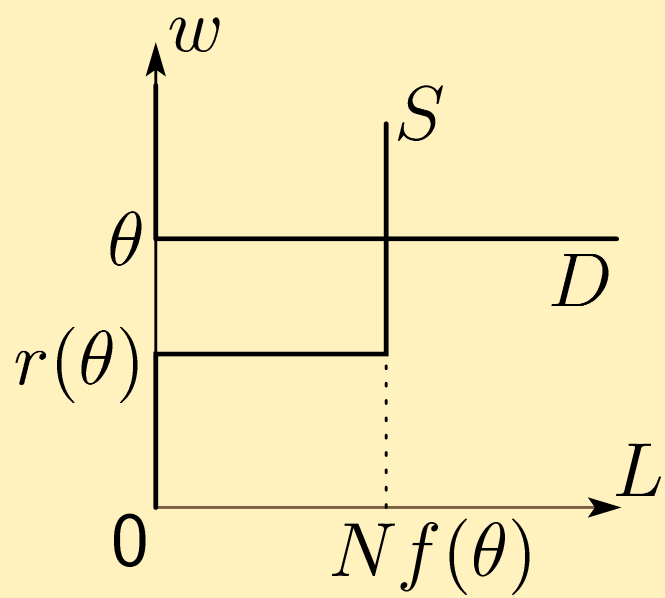

$\Rightarrow$ The market equilibrium maximizes total surplus.

Incomplete information

The wage level $\forall \theta \in \Theta$, $w(\theta) = w$ (constant).

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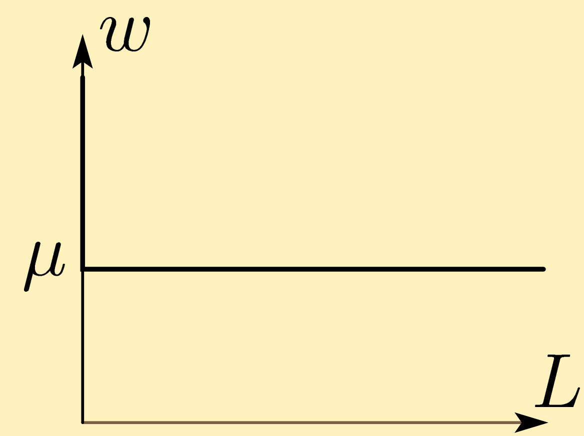
Firms' demand for labor

$$D(w) = \begin{cases} 0 & \text{if } \mu < w, \\ [0, \infty) & \text{if } \mu = w, \\ \infty & \text{if } \mu > w. \end{cases}$$



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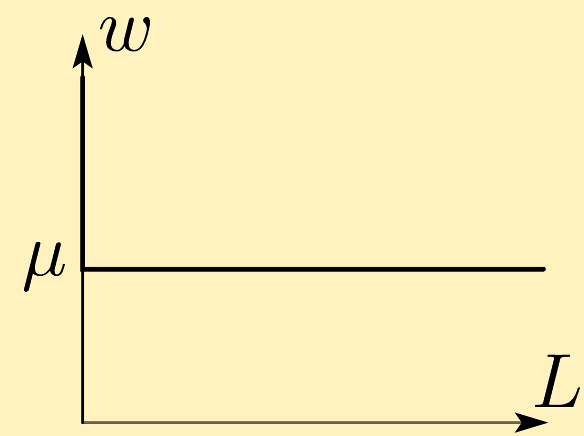

$$\mu = E[\theta | \theta \in \Theta(w)] \equiv \int_{\theta \in \Theta(w)} \theta f(\theta | \theta \in \Theta(w)) d\theta,$$

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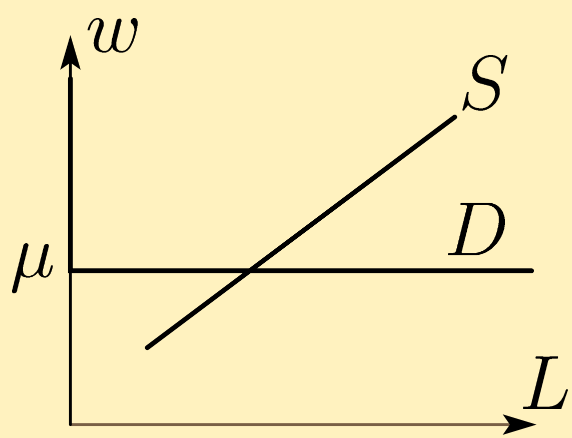
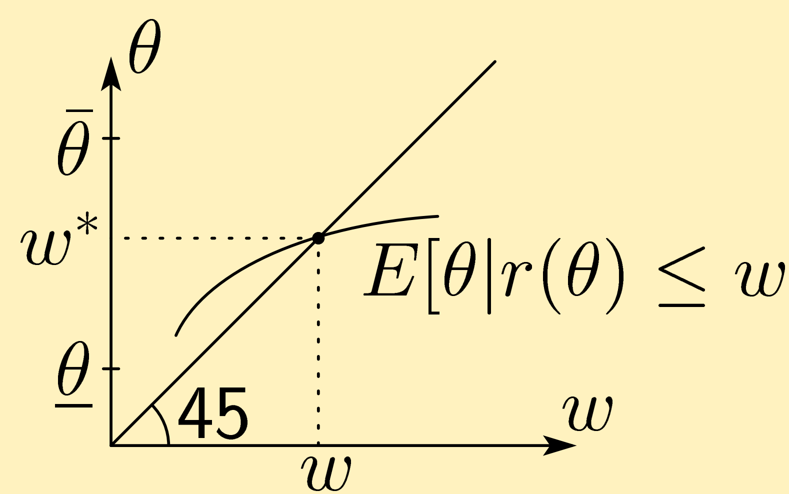
The market equilibrium: $w = \mu = E[\theta|\theta \in \Theta(w)]$.





Incomplete information

Def. In the competitive labor market with unobservable worker productivities, **a competitive equilibrium** is a pair of a wage rate w^* and a set of Θ^* of worker types accepting employment such that

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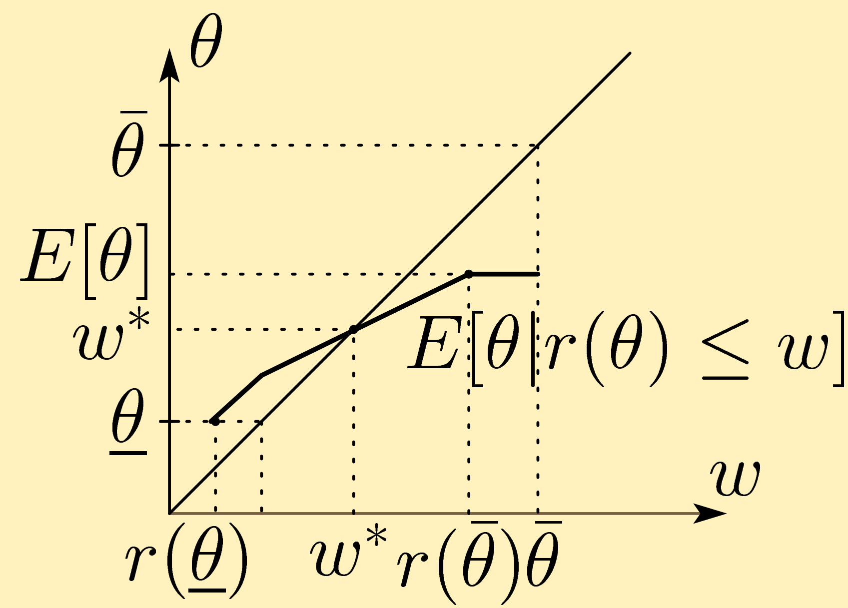
Ex. $r(\theta) = r \ \forall \theta$.

$$\Theta^* = \begin{cases} [\underline{\theta}, \bar{\theta}] & \text{if } E[\theta] \geq r, \\ \phi, & \text{if } E[\theta] < r. \end{cases}$$



Adverse selection

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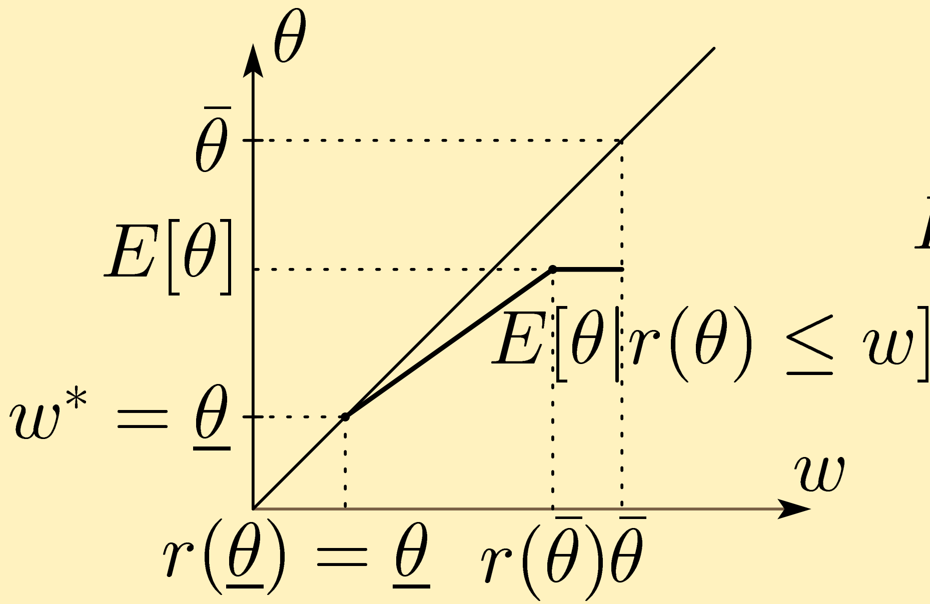

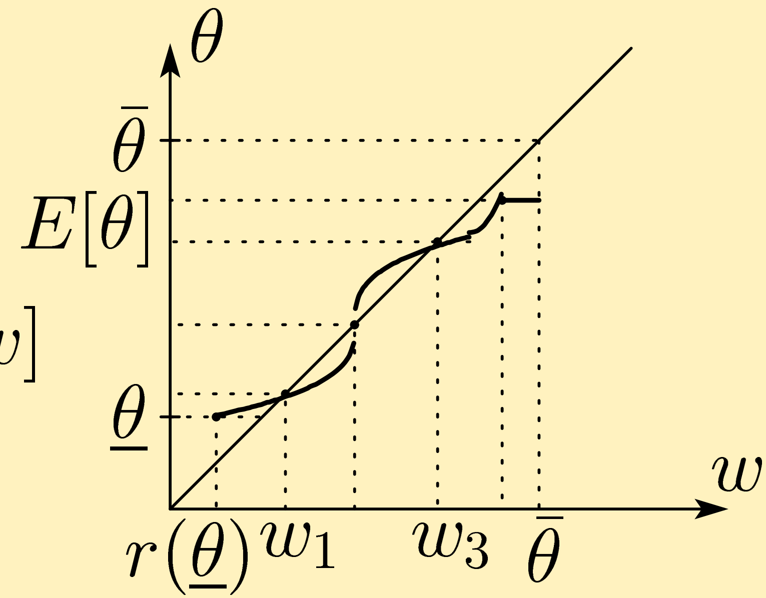

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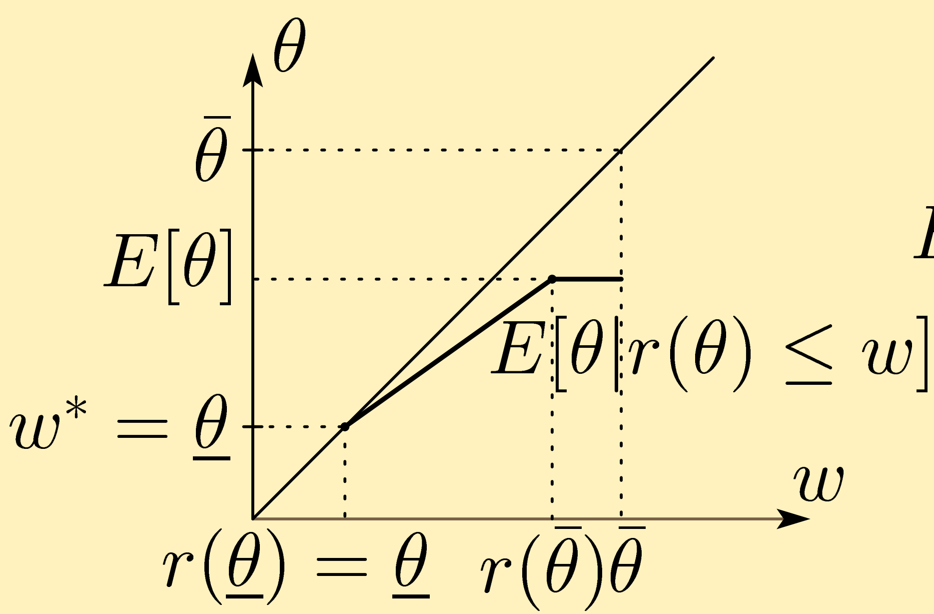

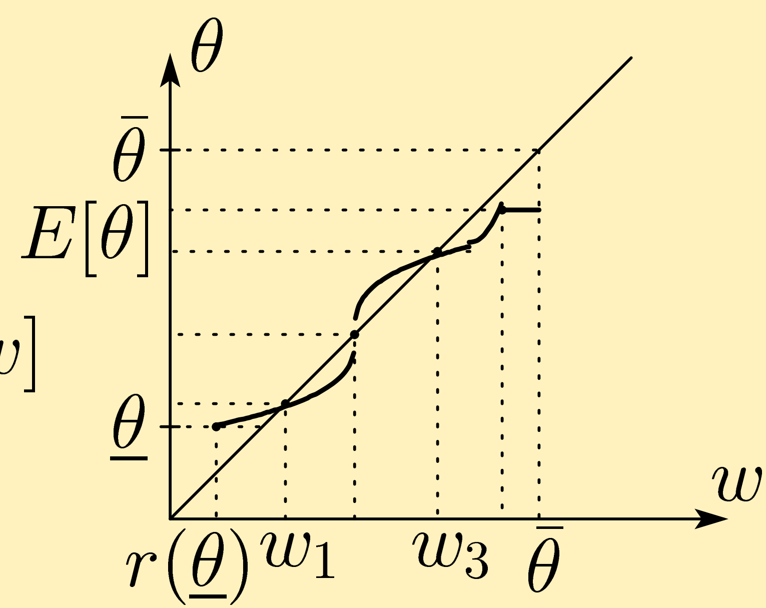

The allocation associated with w_3 Pareto-dominates that with w_1.

Game theoretic approach

Game theoretic approach The following assumptions are imposed: For all $\theta \in \Theta$,

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Game description of two firms

1. Nature determines workers' type following F.

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1. Nature determines workers' type following F.
2. Two firms simultaneously announce their wage offers w_1 and w_2.
3. Workers decide whether to work for a firm and, if so, which one.

Game theoretic approach

Proposition 13.B.1 Let W^* be the set of competitive equilibrium wages, and $w^* = \max W^*$.

Game theoretic approach

Proposition 13.B.1 $w^* = \max W^*$

1. If $w^* > r(\underline{\theta})$ and there is $\varepsilon > 0$ such that $\forall w' \in (w^* - \varepsilon, w^*)$, $E[\theta | r(\theta) \leq w'] > w'$, then a pure strategy SPNE is unique, and in this SPNE, both firms offer w^*, employed workers accept w^*, and the set of employed workers is $\Theta(w^*) \equiv \{\theta | r(\theta) \leq w^*\}$.
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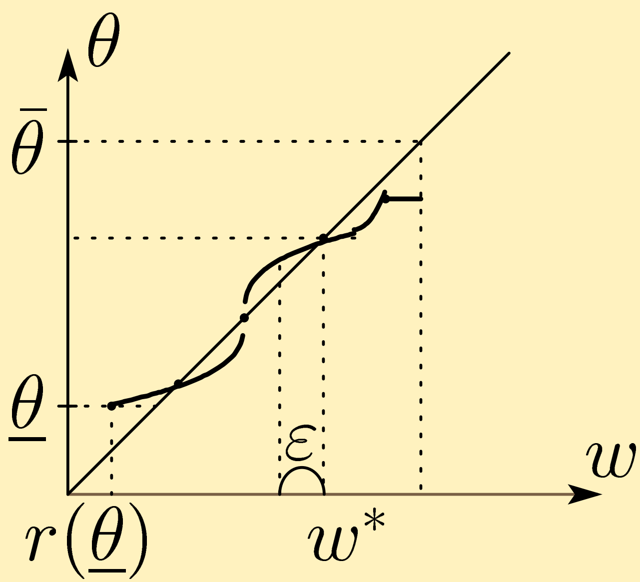

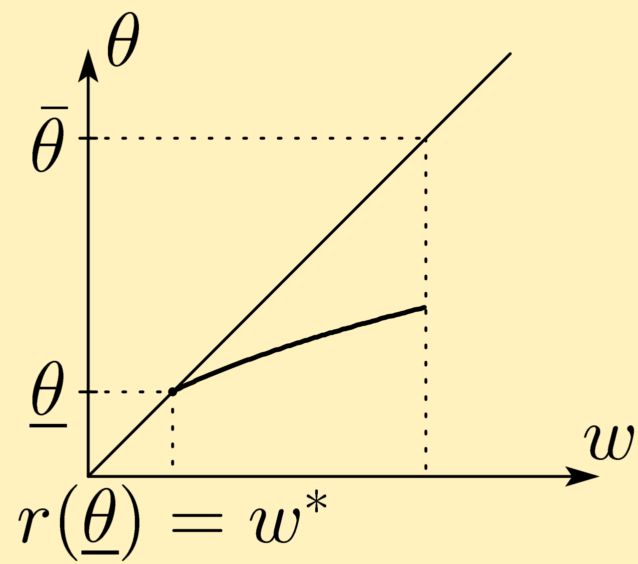

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Proof of Proposition 13.B.1 First, note that, in any SPNE, type θ worker accepts one of the highest wage offer if it is higher than $r(\theta)$, otherwise does not.

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Let $\bar{w}$ be the highest wage at the SPNE. Then,

$$\bar{w} \geq r(\underline{\theta}) \;\; and \;\; \Pi \equiv \pi_1 + \pi_2 = M\{E[\theta | r(\theta) \leq \bar{w}] - \bar{w}\},$$

where M is the number of workers.

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where M is the number of workers. Note that $\pi_1 \leq \Pi/2$ or $\pi_2 \leq \Pi/2$. Suppose that $\pi_1 \leq \Pi/2$ wlog.

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Proof of 1. (Step 1) cont.

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If firm 1 offers $w_1 = \bar{w} + \alpha$ $(\alpha > 0)$, it at least earns

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Hence $\Pi \leq 0$, that is, $\pi_1 = \pi_2 = 0$.

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Proof of 1. (Step 2) w^* is the highest wage in SPNE.

Let $\bar{w}$ be the highest wage in an SPNE.

Game theoretic approach

Proof of 1. (Step 1) cont.

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Since $\pi_i = 0$, $\bar{w} \in W^*$ or $\bar{w} \leq r(\underline{\theta})$.

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Since $\pi_i = 0$, $\bar{w} \in W^*$ or $\bar{w} \leq r(\underline{\theta})$. Suppose that $\bar{w} < w^*$. By offering $w' \in (\max\{w^* - \varepsilon, \bar{w}\}, w^*)$, a firm can earn a positive profit, a contradiction.

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Proof of 1. (Step 3) $w_1 = w_2 = w^*.$

Game theoretic approach

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Game theoretic approach

Proof of 1. (Step 3) $w_1 = w_2 = w^*$.

Let $w_2 < w_1 = w^*$. By offering $w' \in (\max\{w^* - \varepsilon, w_2\}, w^*)$, firm 1 can earn a positive profit. A contradiction.

Game theoretic approach

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As $w \to \infty$, $E[\theta|r(\theta) \leq w] \to E[\theta]$, which is finite.

Therefore, $E[\theta|r(\theta) \leq w] - w < 0$.

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$E[\theta | r(\theta) \leq w] - w < 0$. Offering $w_i > w^*$ is not profitable.

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Thus, $w_1 \leq w^* = r(\underline{\theta})$ and $w_2 \leq w^* = r(\underline{\theta})$.

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Thus, $w_1 \leq w^* = r(\underline{\theta})$ and $w_2 \leq w^* = r(\underline{\theta})$.

If firm i offers $w_i \leq w^*$, no worker accepts w_i, and then its profit is zero.

The set of wage offer (w_1, w_2) in SPNE is

$$\{(w_1, w_2) | w_i \leq w^*, i = 1, 2\}.$$