# Chapter §9.C in MWG: Beliefs and Sequential Rationality
# Chapter 15 in Tadelis: Sequential Rationality with Incomplete Information

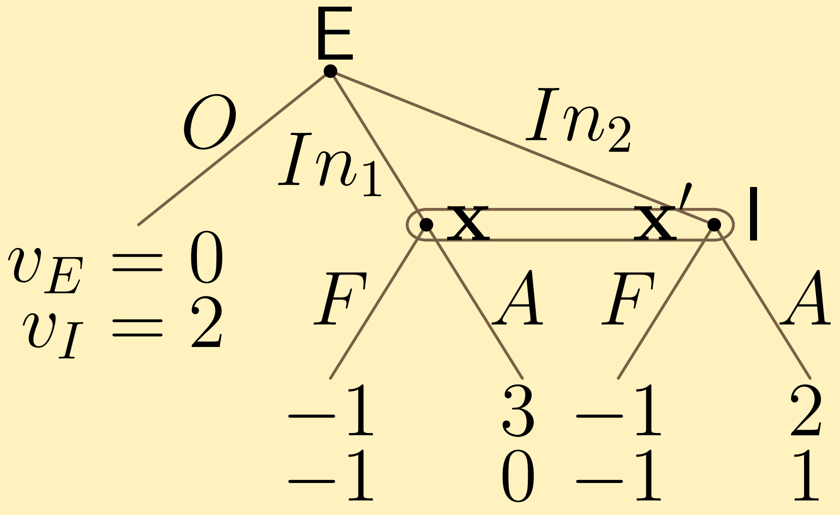
- The problem with subgame perfection
- Perfect Bayesian Equilibrium
- Sequential Equilibrium

## 15.1 The problem with subgame perfection

## §9.C Beliefs and Sequential Rationality

**Example 9.C.1** No subgame except for the whole game.

| E/I | $F$ | $A$ |
|---|---|---|
| $O$ | $\mathbf{0, 2}$ | $0, 2$ |
| $In_1$ | $-1, -1$ | $\mathbf{3, 0}$ |
| $In_2$ | $-1, -1$ | $2, 1$ |

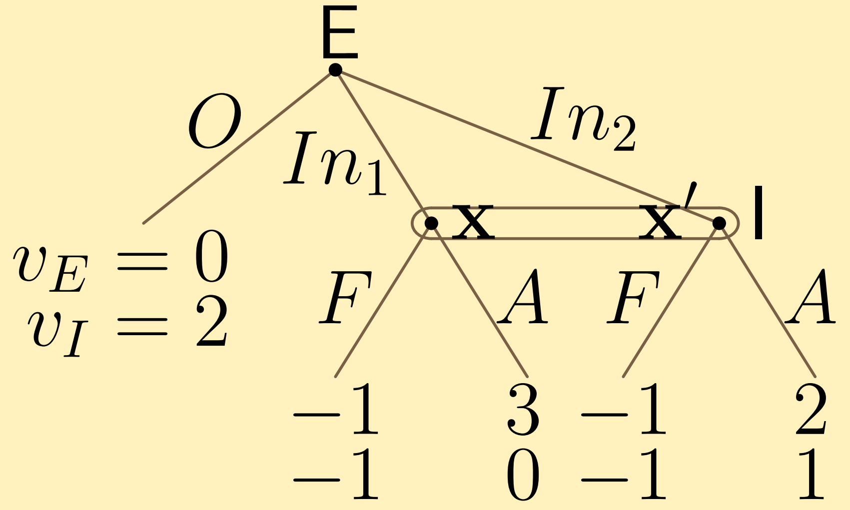



There are two subgame perfect Nash equilibria.

## §9.C Beliefs and Sequential Rationality

**Example 9.C.1** No subgame except for the whole game.

| E/I | $F$ | $A$ |
|---|---|---|
| $O$ | $\mathbf{0, 2}$ | $0, 2$ |
| $In_1$ | $-1, -1$ | $\mathbf{3, 0}$ |
| $In_2$ | $-1, -1$ | $2, 1$ |

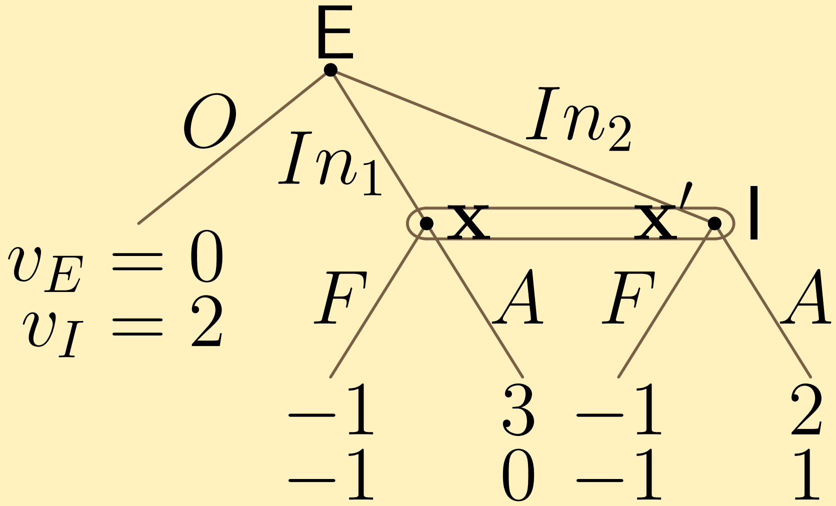



There are two subgame perfect Nash equilibria. However, once player E enters, whether plays $In_1$ or $In_2$, it is optimal for player I to play $A$.

## §9.C Beliefs and Sequential Rationality

**Example 9.C.1** No subgame except for the whole game.

| E/I | $F$ | $A$ |
|---|---|---|
| $O$ | **0, 2** | $0, 2$ |
| $In_1$ | $-1, -1$ | **3, 0** |
| $In_2$ | $-1, -1$ | $2, 1$ |




There are two subgame perfect Nash equilibria. However, once player E enters, whether plays $In_1$ or $In_2$, it is optimal for player I to play $A$. Thus, $(O, F)$ is NOT consistent with the spirit of sequential rationality.

## 15.2 Perfect Bayesian Equilibrium

**Definition 15.1** Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies in a game of incomplete information.

An information set is **on the equilibrium path** if given $\sigma^*$ and the distribution of types, it is reached *with positive probability*.

An information set is **off the equilibrium path** if given $\sigma^*$ and the distribution of types, it is reached *with zero probability*.

As in Chapter 7, we define the two concepts here.

**Definition 15.2** A **system of beliefs** $\mu$ of an extensive-form game assigns a probability distribution over decision nodes to every information set.

That is, for every information set $h \in H$ and every decision node $x \in h$, $\mu(x) \in [0, 1]$ is the probability that player $i$ who moves in information set $h$ assigns to his being at $x$, where

$$\sum_{x \in h} \mu(x) = 1,$$

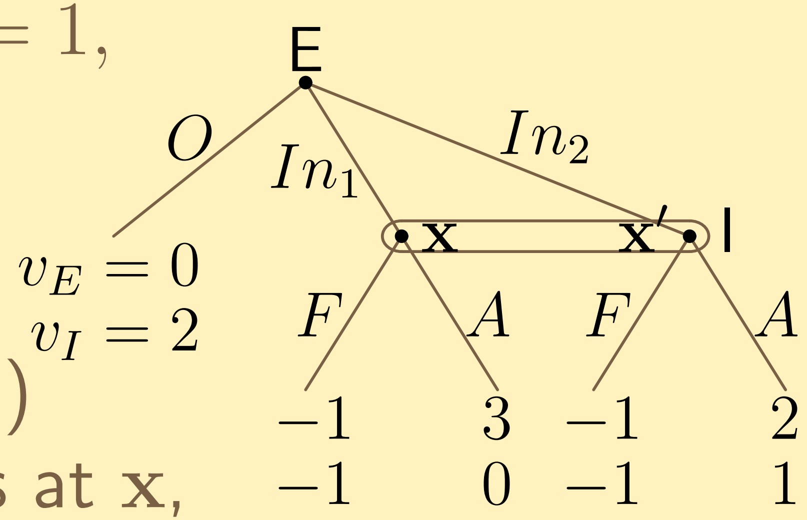
for every information set $h \in H$.

**Definition 15.2**

$\mu(x) \in [0,1]$ is the probability that player $i$ who moves in information set $h$ assigns to his being at $x$, where

$$\sum_{x \in h} \mu(x) = 1,$$

for every information set $h \in H$.




Player $I$'s belief $(\mu(\mathbf{x}) + \mu(\mathbf{x}') = 1)$

$\mu(\mathbf{x})$: Player $I$'s belief that he is at $\mathbf{x}$,

$\mu(\mathbf{x}')$: Player $I$'s belief that he is at $\mathbf{x}'$.

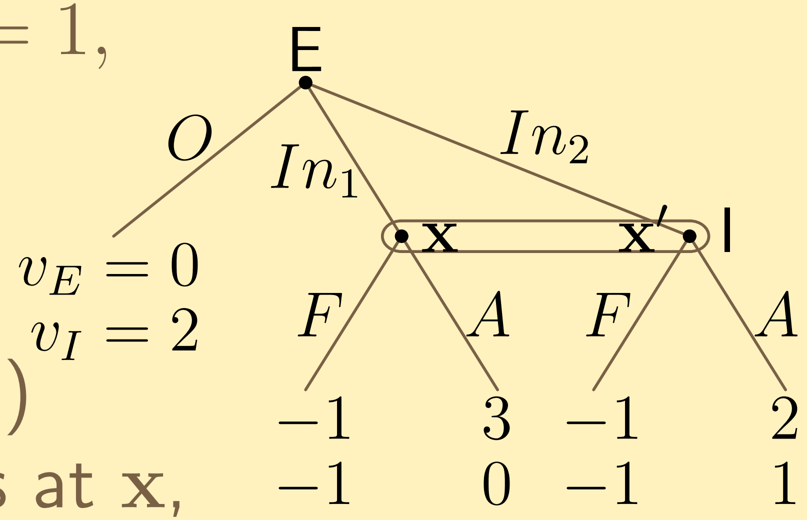
# Beliefs and Sequential Rationality

**Definition 15.2**

$\mu(x) \in [0,1]$ is the probability that player $i$ who moves in information set $h$ assigns to his being at $x$, where

$$\sum_{x \in h} \mu(x) = 1,$$

for every information set $h \in H$.




Player $I$'s belief $(\mu(\mathbf{x}) + \mu(\mathbf{x}') = 1)$

$\mu(\mathbf{x})$: Player $I$'s belief that he is at $\mathbf{x}$,

$\mu(\mathbf{x}')$: Player $I$'s belief that he is at $\mathbf{x}'$.

For any $\mu(\mathbf{x})$, play $A$ is better for player $I$ than play $F$.

Calculate its expected payoffs of play $A$ and play $F$.

# Beliefs and Sequential Rationality

**Requirement 15.1** Every player will have a well-defined belief over where he is in each of his information sets. That is, the game will have a *system of beliefs*.

How do we determine the beliefs?

# Beliefs and Sequential Rationality

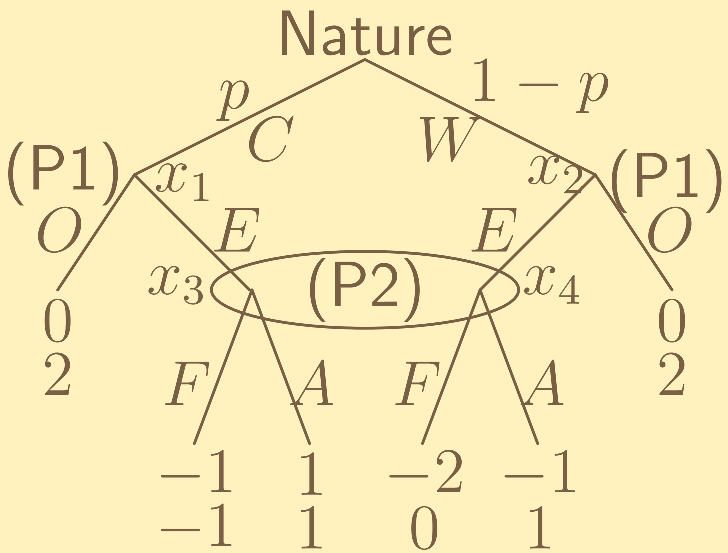
**Requirement 15.1** The game will have a *system of beliefs*.

If P1's strategy is $EO$, $\mu(x_3) = 1$.

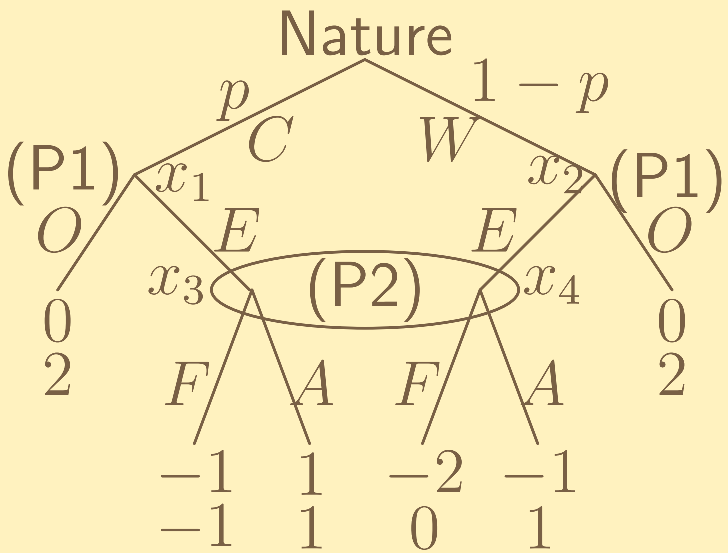
$x_3$ is reached with prob. $p \times 1$

$x_4$ is reached with prob. $(1-p)0$.

Applying Bayes' rule, we have $\mu(x_3) = 1$.



**Requirement 15.1** The game will have a *system of beliefs*.




If P1's strategy is $EO$, $\mu(x_3) = 1$.
By considering behavioral strategy, we generalize the argument.

If P1's strategy is $(\sigma_C, \sigma_W)$, where $\sigma_k$ is the probability that he plays $E$ when he is $k$ $(k = C, W)$,

$$\mu(x_3) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$

Here, we ignore the case in which $(\sigma_C, \sigma_W) = (0, 0)$ (that is, $OO$). Explain it later.

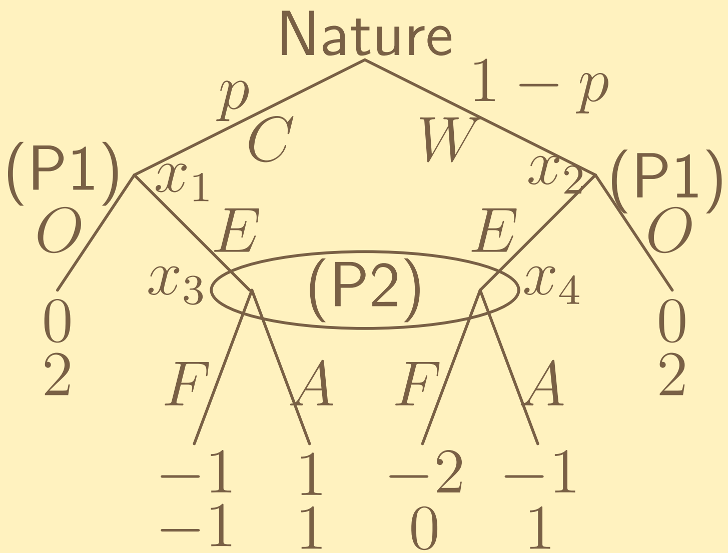
# Beliefs and Sequential Rationality

**Requirement 15.1** The game will have a *system of beliefs*.

If P1's strategy is $EO$, $\mu(x_3) = 1$.

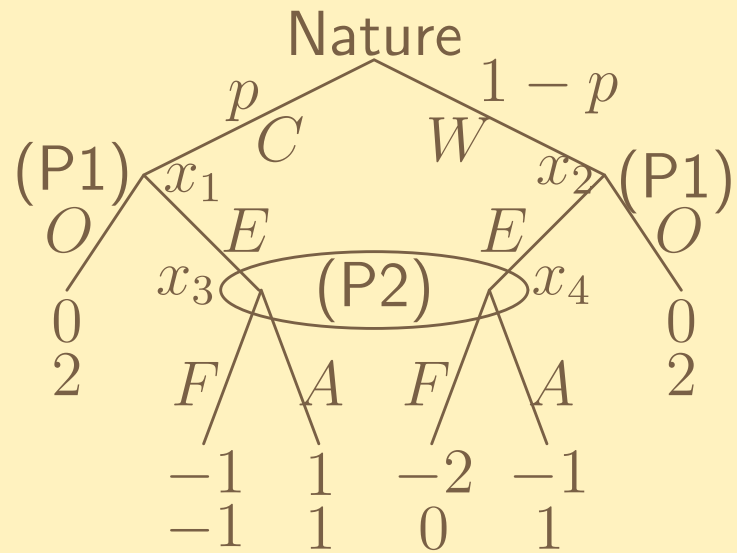
If P1's strategy is $(\sigma_C, \sigma_W)$, where $\sigma_k$ is the probability that he plays $E$ when he is $k$ ($k = C, W$),

$$\mu(x_3) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$




**Requirement 15.2** Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies. We require that in all information sets, beliefs that are on the equilibrium path be consistent with *Bayes' rule*.

**Requirement 15.1** The game will have a *system of beliefs*.




If P1's strategy is $(\sigma_C, \sigma_W)$, where $\sigma_k$ is the probability that he plays $E$ when he is $k$ $(k = C, W)$,

$$\mu(x_3) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$

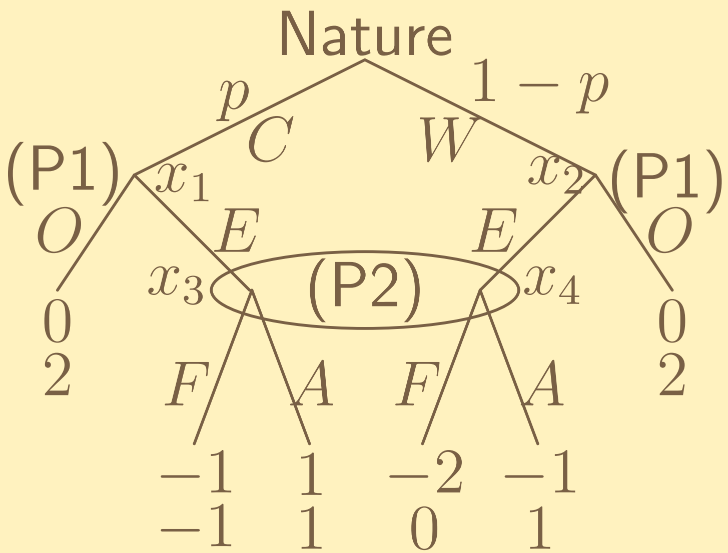
If P1's strategy is $(\sigma_C, \sigma_W) = (0, 0)$ (that is, $OO$), the information set of P2 is off the equilibrium path.

# Beliefs and Sequential Rationality

**Requirement 15.1** The game will have a *system of beliefs*.

If P1's strategy is $(\sigma_C, \sigma_W)$, where $\sigma_k$ is the probability that he plays $E$ when he is $k$ $(k = C, W)$,

$$\mu(x_3) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$




If P1's strategy is $(\sigma_C, \sigma_W) = (0, 0)$ (that is, $OO$), the information set of P2 is off the equilibrium path.

**Requirement 15.3** At information sets that are off the equilibrium path, any belief can be assigned.

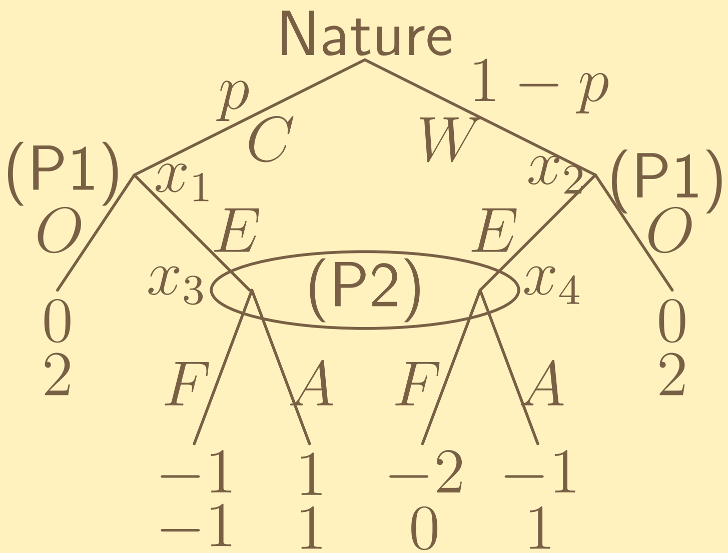
We must assign beliefs even for such information sets.

# Beliefs and Sequential Rationality

**Requirement 15.1** The game will have a *system of beliefs*.

If P1's strategy is $(\sigma_C, \sigma_W)$, where $\sigma_k$ is the probability that he plays $E$ when he is $k$ $(k = C, W)$,

$$\mu(x_3) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$




**Requirement 15.2** In all information sets beliefs that are on the equilibrium path be consistent with *Bayes' rule*.

**Requirement 15.3** At information sets that are off the equilibrium path, any belief can be assigned.

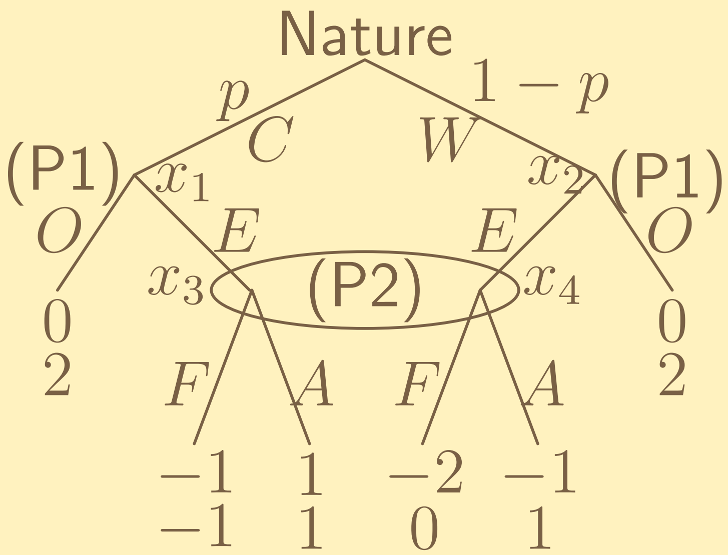
Those define how to assign beliefs in information sets.

# Beliefs and Sequential Rationality

**Requirement 15.1** The game will have a *system of beliefs*.

If P1's strategy is $(\sigma_C, \sigma_W)$, where $\sigma_k$ is the probability that he plays $E$ when he is $k$ $(k = C, W)$,

$$\mu(x_3) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$




**Requirement 15.2** In all information sets beliefs that are on the equilibrium path be consistent with *Bayes' rule*.

**Requirement 15.4** Given their beliefs, players' strategies must be sequentially rational. That is, in every information set, players will play a best response to their beliefs.

# Beliefs and Sequential Rationality

**Def. Expected utility** $E[v_i(\sigma_i, \sigma_{-i}, \theta_i)|h, \mu]$

Player $i$'s expected payoff starting at his information set $h$ if the belief is given by $\mu$, if player $i$ follows $\sigma_i$ and others follow $\sigma_{-i}$, and if his type is $\theta_i$.

$\iota(h)$: the player who moves at information set $h$.

**Def. 9.C.2** A strategy profile $\sigma$ of an extensive-form game is **sequentially rational** at information set $h$ given a system of beliefs $\mu$ if $\forall \hat{\sigma}_{\iota(h)} \in \Delta(S_{\iota(h)})$,

$$\begin{aligned} & E[v_{\iota(h)}(\sigma_{\iota(h)}, \sigma_{-\iota(h)}, \theta_i)|h, \mu] \\ \geq \ & E[v_{\iota(h)}(\hat{\sigma}_{\iota(h)}, \sigma_{-\iota(h)}, \theta_i)|h, \mu]. \end{aligned}$$

# Beliefs and Sequential Rationality

**Def. Expected utility** $E[v_i(\sigma_i, \sigma_{-i}, \theta_i)|h, \mu]$

$\iota(h)$: the player who moves at information set $h$.

**Def. 9.C.2** A strategy profile $\sigma$ of an extensive-form game is **sequentially rational** at information set $h$ given a system of beliefs $\mu$ if $\forall \hat{\sigma}_{\iota(h)} \in \Delta(S_{\iota(h)})$,

$$\begin{aligned} & E[v_{\iota(h)}(\sigma_{\iota(h)}, \sigma_{-\iota(h)}, \theta_i)|h, \mu] \\ \geq\ & E[v_{\iota(h)}(\hat{\sigma}_{\iota(h)}, \sigma_{-\iota(h)}, \theta_i)|h, \mu]. \end{aligned}$$

A strategy profile $\sigma$ is sequentially rational given $\mu$, if $\sigma$ is sequentially rational at any information set $h$ given $\mu$.

# Update of Beliefs

$\Pr(x|\sigma)$**:** Prob. of reaching a node $x$ given play of $\sigma$.

# Update of Beliefs

$\Pr(x|\sigma)$**:** Prob. of reaching a node $x$ given play of $\sigma$.

$\Pr(h|\sigma) = \sum_{x \in h} \Pr(x|\sigma)$**:** Prob. of reaching an information set $h$ given play of $\sigma$.

# Update of Beliefs

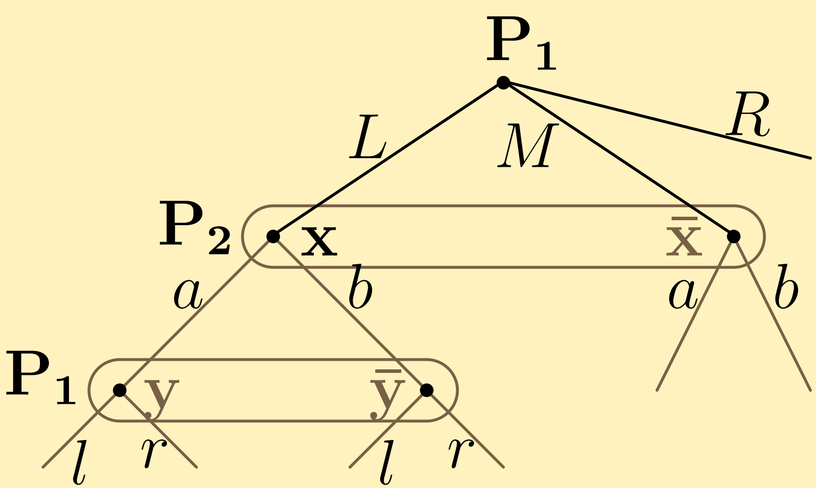
$\Pr(x|\sigma)$: Prob. of reaching a node $x$ given play of $\sigma$.

$\Pr(h|\sigma) = \sum_{x \in h} \Pr(x|\sigma)$: Prob. of reaching an information set $h$ given play of $\sigma$.

$\Pr(x|h, \sigma) = \dfrac{\Pr(x|\sigma)}{\sum_{x' \in h} \Pr(x'|\sigma)}$: Cond. Prob. of being at $x$ given that $\sigma$ and the play reached $h$.

# Update of Beliefs

$\Pr(x|h,\sigma) = \dfrac{\Pr(x|\sigma)}{\sum_{x'\in h}\Pr(x'|\sigma)}$: Cond. Prob. of being at $x$ given that $\sigma$ and the play reached $h$.




$$\begin{aligned}\Pr(y|h_y,\sigma) &= \frac{\Pr(y|\sigma)}{\Pr(h_y|\sigma)} \\ &= \frac{\sigma_1(L)\sigma_2(a)}{\sigma_1(L)\sigma_2(a)+\sigma_1(L)\sigma_2(b)} \\ &= \frac{\sigma_2(a)}{\sigma_2(a)+\sigma_2(b)}.\end{aligned}$$

# (weak) Perfect Bayesian Equilibrium

**Def.** A system of beliefs $\mu$ is (weakly) consistent with $\sigma$ if for all information set $h$ with $\Pr(h|\sigma) > 0$ and all $x \in h$, $\mu(x) = \Pr(x|h, \sigma)$.

# (weak) Perfect Bayesian Equilibrium

**Def.** A system of beliefs $\mu$ is (weakly) consistent with $\sigma$ if for all information set $h$ with $\Pr(h|\sigma) > 0$ and all $x \in h$, $\mu(x) = \Pr(x|h, \sigma)$.

**Definition 15.3** A Bayesian Nash equilibrium profile $\sigma^*$ together with a system of beliefs $\mu$ constitutes a (weak) perfect Bayesian equilibrium for an $n$-player game if

1. $\sigma^*$ is sequentially rational given $\mu$ (Req. 15.4).
2. $\mu$ is weakly consistent with $\sigma^*$ (Req. 15.1-3).

# (weak) Perfect Bayesian Equilibrium

**Definition 15.3** A Bayesian Nash equilibrium profile $\sigma^*$ together with a system of beliefs $\mu$ constitutes a (weak) perfect Bayesian equilibrium for an $n$-player game if

1. $\sigma^*$ is sequentially rational given $\mu$ (Req. 15.4).
2. $\mu$ is weakly consistent with $\sigma^*$ (Req. 15.1-3).

**Prop. 9.C.1** A strategy profile $\sigma$ is a Nash equilibrium of $\Gamma$ iff there is a system of beliefs $\mu$ such that

1. $\sigma$ is sequentially rational given $\mu$ at all information sets $h$ such that $\Pr(h|\sigma) > 0$,
   Nash Eq. does NOT require sequential rationality for information sets which are off the equilibrium path.
2. $\mu$ is weakly consistent with $\sigma$.

# (weak) Perfect Bayesian Equilibrium

**Definition 15.3** A Bayesian Nash equilibrium profile $\sigma^*$ together with a system of beliefs $\mu$ constitutes a (weak) perfect Bayesian equilibrium for an $n$-player game if

1. $\sigma^*$ is sequentially rational given $\mu$ (Req. 15.4).
2. $\mu$ is weakly consistent with $\sigma^*$ (Req. 15.1-3).

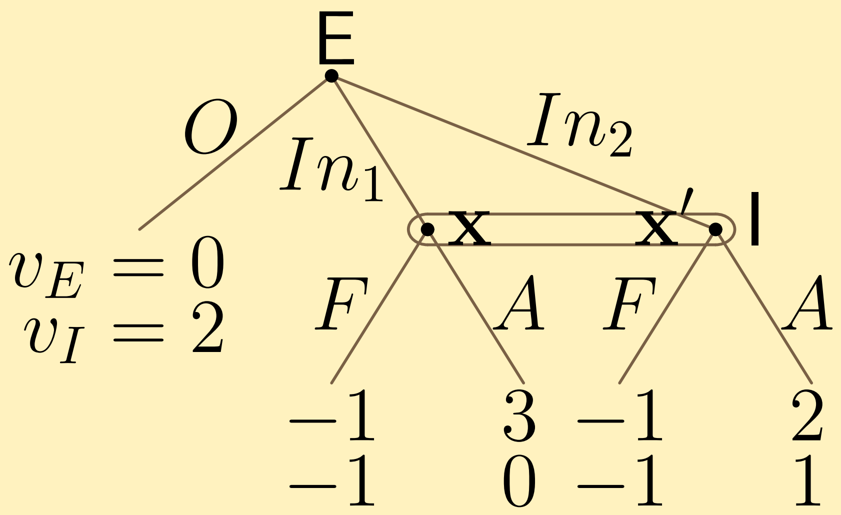
**Prop. 15.1** If a strategy profile $\sigma^*$ is a Bayesian Nash equilibrium of a Bayesian game $\Gamma$, and if $\sigma^*$ induces all the information sets to be reached with positive probability, then $\sigma^*$, together with the belief system $\mu^*$ uniquely derived from $\sigma^*$ and the distribution of types, constitutes a perfect Bayesian equilibrium for $\Gamma$.

$\Rightarrow$ In all the information sets, $\sigma^*$ and $\mu$ satisfy the Reqs.

# Example

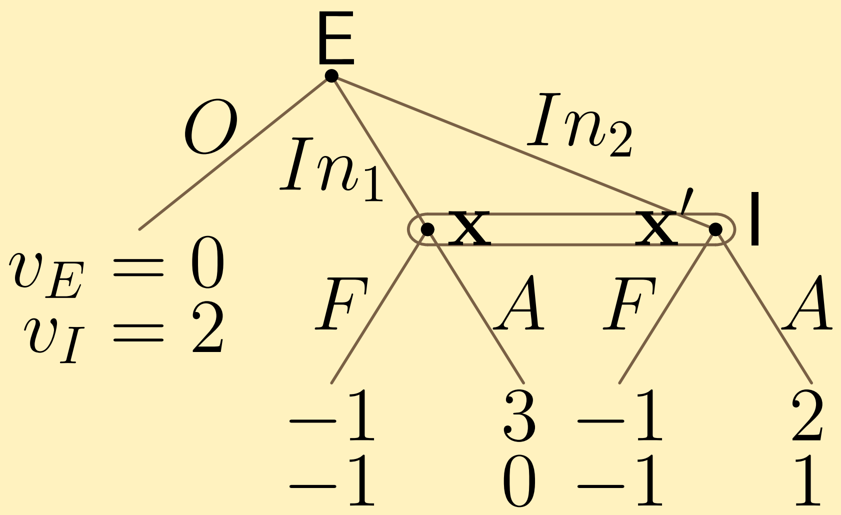
**Example 9.C.1** Since $A$ is a strictly dominant strategy, whatever $\mu$ is, only $A$ is sequentially rational. Nash eq. $(O, F)$ is NOT a weakly perfect Bayesian equilibrium.

| E/I | $F$ | $A$ |
|---|---|---|
| $O$ | $\mathbf{0,2}$ | $0,2$ |
| $In_1$ | $-1,-1$ | $\mathbf{3,0}$ |
| $In_2$ | $-1,-1$ | $2,1$ |



**Example 9.C.1** Since $A$ is a strictly dominant strategy, whatever $\mu$ is, only $A$ is sequentially rational. Nash eq. $(O, F)$ is NOT a weakly perfect Bayesian equilibrium.

| E/I | $F$ | $A$ |
|---|---|---|
| $O$ | $\mathbf{0, 2}$ | $0, 2$ |
| $In_1$ | $-1, -1$ | $\mathbf{3, 0}$ |
| $In_2$ | $-1, -1$ | $2, 1$ |

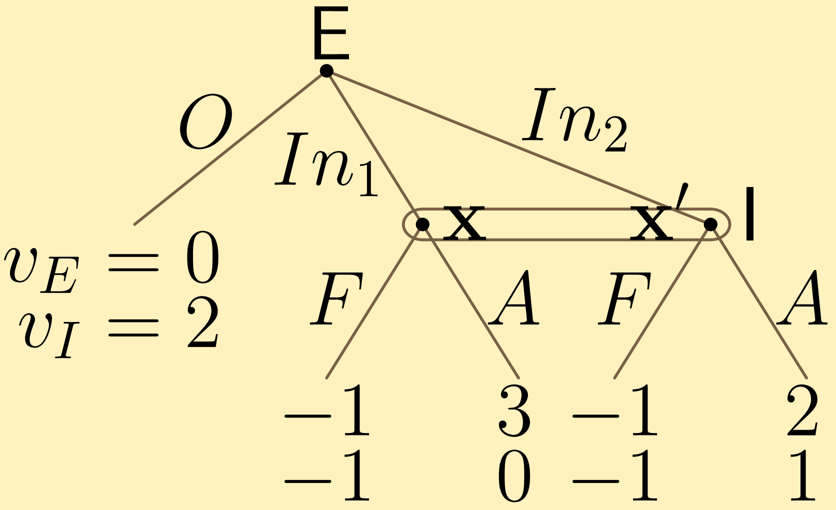



Given that $E$ chooses $\sigma(O) = 1$, there is NO requirement for the belief of $I$.

# Example

**Example 9.C.1** Since $A$ is a strictly dominant strategy, whatever $\mu$ is, only $A$ is sequentially rational. Nash eq. $(O, F)$ is NOT a weakly perfect Bayesian equilibrium.

| E/I | $F$ | $A$ |
|---|---|---|
| $O$ | $\mathbf{0, 2}$ | $0, 2$ |
| $In_1$ | $-1, -1$ | $\mathbf{3, 0}$ |
| $In_2$ | $-1, -1$ | $2, 1$ |

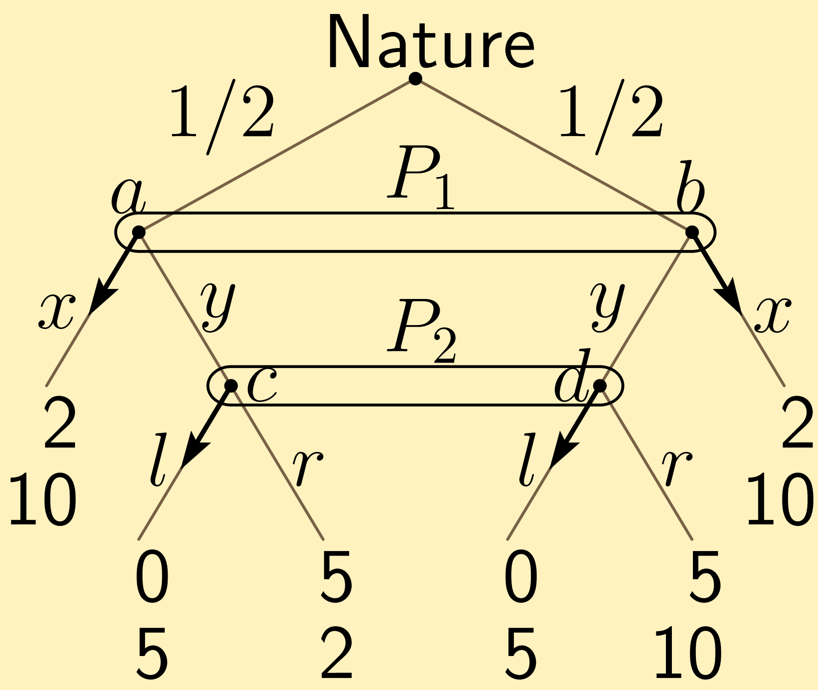



Given that $E$ chooses $\sigma(O) = 1$, there is NO requirement for the belief of $I$. However, for any $\mu$, $A$ is a strictly dominant strategy.

## Strengthening the wPBE concept (9.C.4)

$$\mu(a) = 1/2,\ \mu(b) = 1/2,\ \mu(c) = 9/10,\ \mu(d) = 1/10.$$

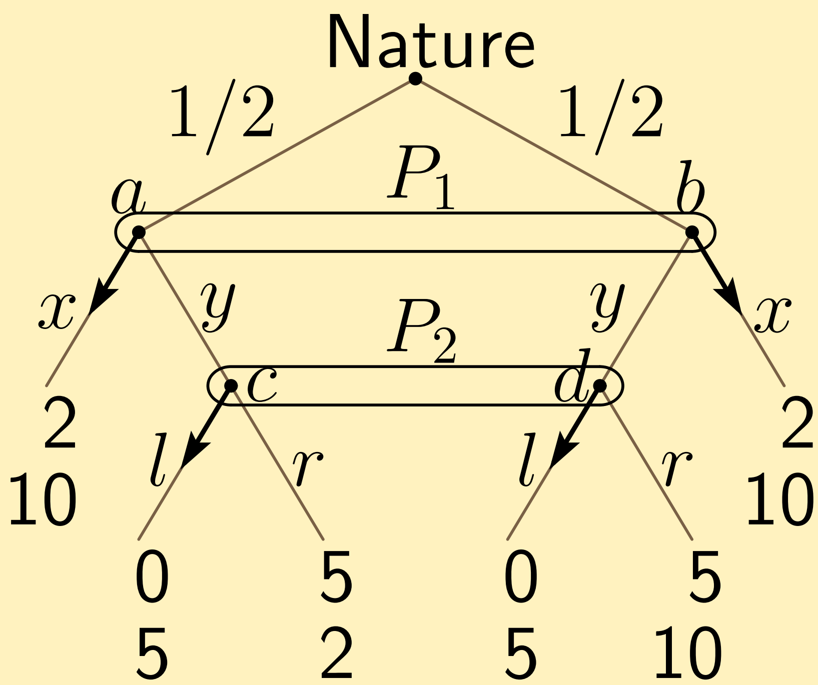
The arrows in the figure indicate the strategies.



## Strengthening the wPBE concept (9.C.4)

$$\mu(a) = 1/2,\ \mu(b) = 1/2,\ \mu(c) = 9/10,\ \mu(d) = 1/10.$$

The arrows in the figure indicate the strategies.




No restrictions at all are placed on beliefs **off the equilibrium path** (see Req. 15-3).

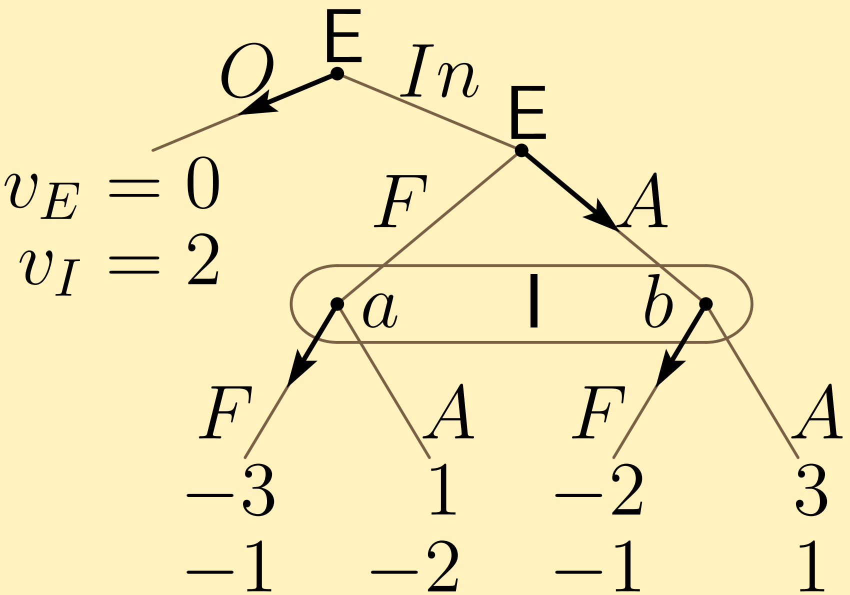
However, $\mu(c) = 9/10$ seems to be structurally inconsistent.

$\mu(c) = 1/2$ would be more reasonable (see Nature's choice).

## Strengthening the wPBE concept (9.C.5)

$$\mu(a) = 1,\ \mu(b) = 0.$$

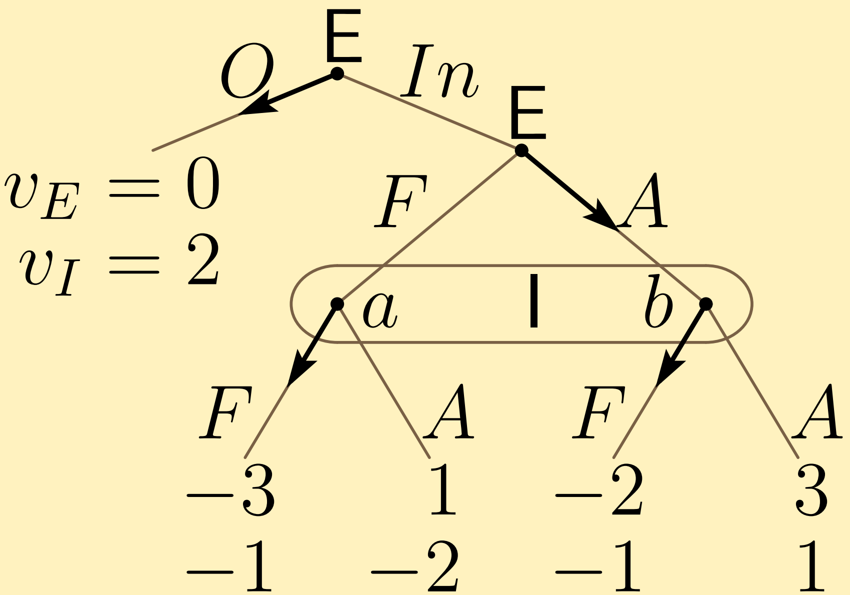
The arrows in the figure indicate the strategies.



## Strengthening the wPBE concept (9.C.5)

$$\mu(a) = 1,\ \mu(b) = 0.$$

The arrows in the figure indicate the strategies.



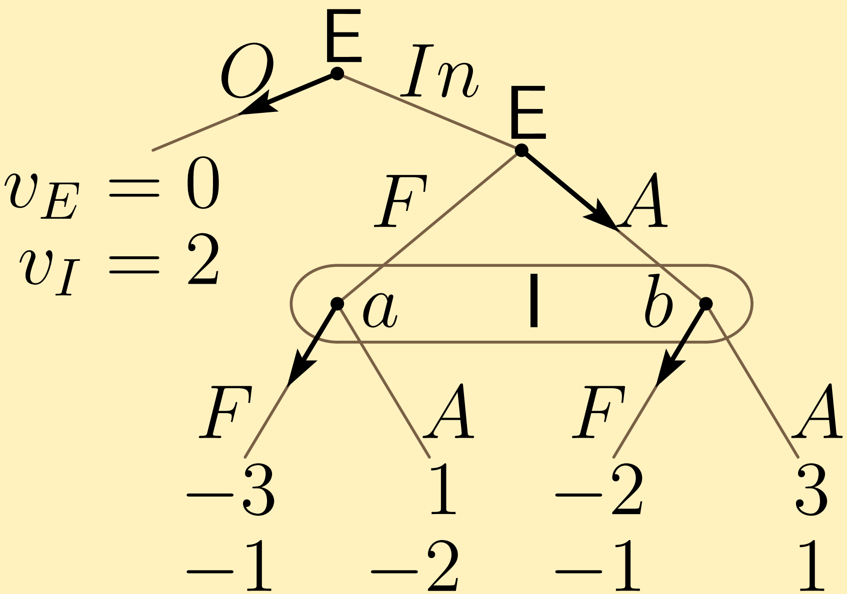

| E/I | $F$ | $A$ |
|---|---|---|
| $F$ | $-3, \mathbf{-1}$ | $1, -2$ |
| $A$ | $\mathbf{-2}, -1$ | $\mathbf{3}, \mathbf{1}$ |

No restrictions at all are placed on beliefs **off the equilibrium path**.

## Strengthening the wPBE concept (9.C.5)

$$\mu(a) = 1, \ \mu(b) = 0.$$

The arrows in the figure indicate the strategies.




| E/I | $F$ | $A$ |
|---|---|---|
| $F$ | $-3, \mathbf{-1}$ | $1, -2$ |
| $A$ | $\mathbf{-2}, -1$ | $\mathbf{3}, \mathbf{1}$ |

No restrictions at all are placed on beliefs **off the equilibrium path**.

This outcome is NOT a SPNE outcome.

# Sequential equilibrium

**Definition 15.4** A system of beliefs $\mu$ is consistent with $\sigma$ if there is a sequence $\{\sigma^k\}_{k=1}^{\infty}$ of total mixed strategies such that

1. $\lim_{k\to\infty} \sigma^k = \sigma$, 2. $\mu = \lim_{k\to\infty} \mu^k$,

where for all $k$, $\mu^k$ is the system of beliefs derived from $\sigma^k$ by Bayes' rule.

# Sequential equilibrium

**Definition 15.4** A system of beliefs $\mu$ is consistent with $\sigma$ if there is a sequence $\{\sigma^k\}_{k=1}^{\infty}$ of total mixed strategies such that

1. $\lim_{k\to\infty} \sigma^k = \sigma$, 2. $\mu = \lim_{k\to\infty} \mu^k$,

where for all $k$, $\mu^k$ is the system of beliefs derived from $\sigma^k$ by Bayes' rule.

**Definition 15.5** A pair $(\sigma, \mu)$ of strategy profile and system of beliefs is a sequential equilibrium if

1. $\sigma$ is sequential rational given $\mu$.
2. $\mu$ is consistent with $\sigma$.

# Sequential equilibrium

**Definition 15.4** A system of beliefs $\mu$ is consistent with $\sigma$ if there is a sequence $\{\sigma^k\}_{k=1}^{\infty}$ of total mixed strategies such that

1. $\lim_{k\to\infty} \sigma^k = \sigma$, 2. $\mu = \lim_{k\to\infty} \mu^k$,

**Definition 15.5** A pair $(\sigma, \mu)$ of strategy profile and system of beliefs is a sequential equilibrium if

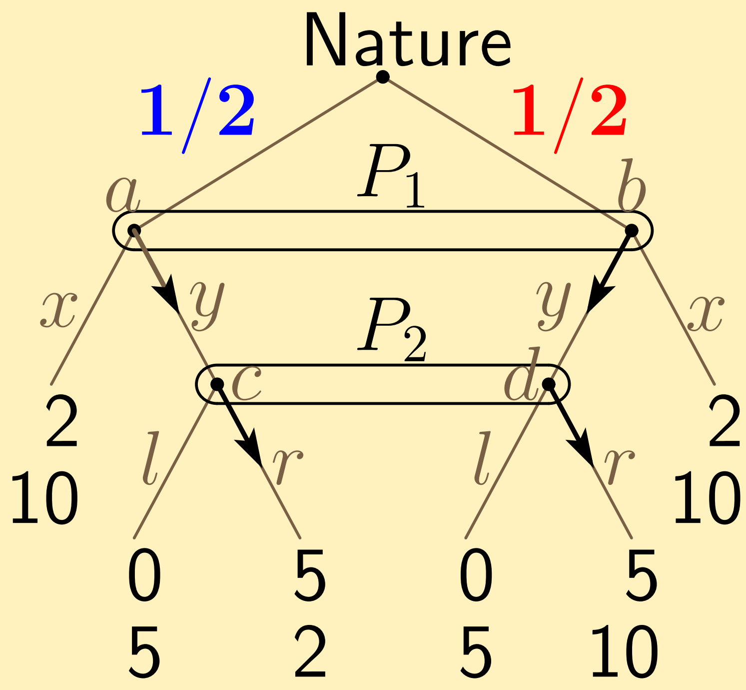
1. $\sigma$ is sequential rational given $\mu$.
2. $\mu$ is consistent with $\sigma$.

**Prop. 9.C.2** In every sequential equilibrium $(\sigma, \mu)$ of $\Gamma_E$, a strategy profile $\sigma$ is a subgame perfect Nash equilibrium of $\Gamma_E$

## Ex. 9.C.4 Reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

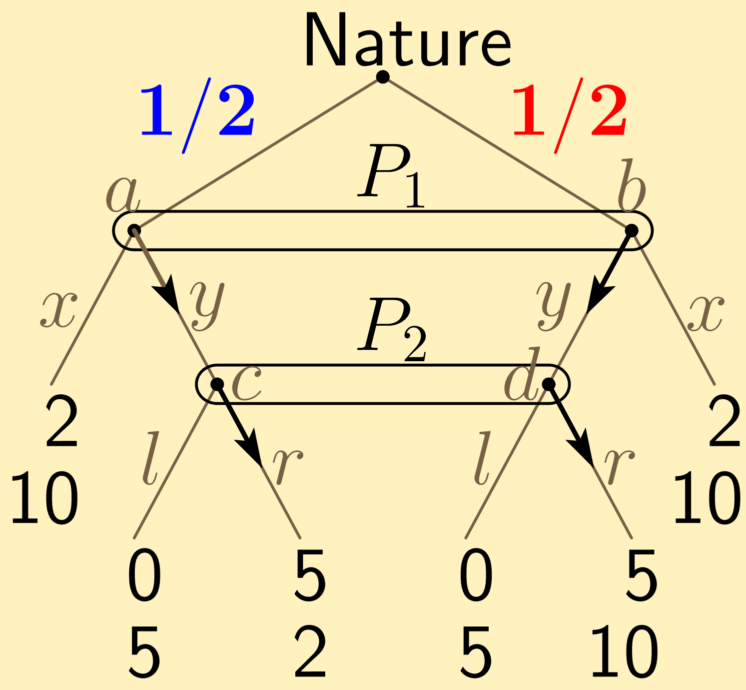
$$\forall k,\ \mu^k(a) = \frac{1}{2} \text{ and } \mu^k(c) = \frac{0.5\sigma_1^k(y)}{0.5\sigma_1^k(y) + 0.5\sigma_1^k(y)} = \frac{1}{2}.$$



## Ex. 9.C.4 Reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{1}{2} \text{ and } \mu^k(c) = \frac{0.5\sigma_1^k(y)}{0.5\sigma_1^k(y) + 0.5\sigma_1^k(y)} = \frac{1}{2}.$$

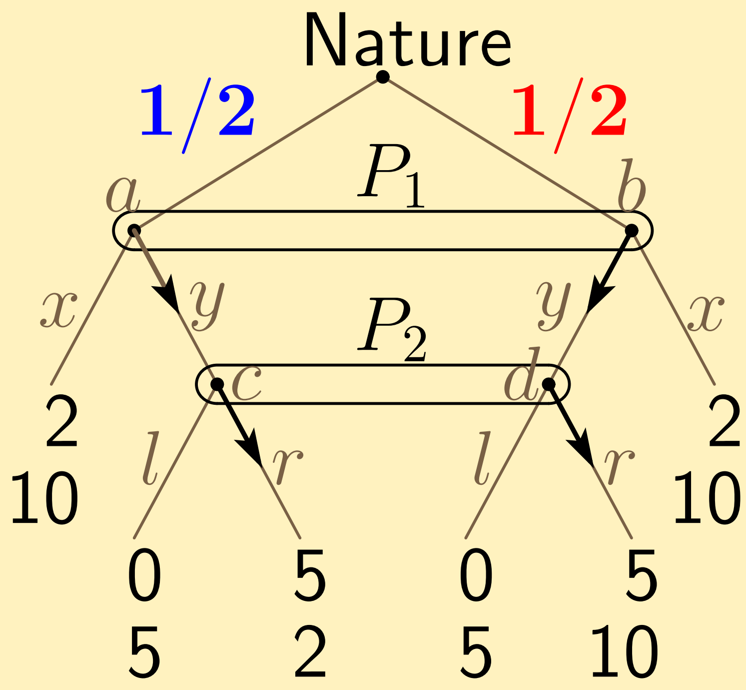



$\lim_{k\to\infty} \mu^k(a) = 1/2,$
$\lim_{k\to\infty} \mu^k(c) = 1/2.$

## Ex. 9.C.4 Reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{1}{2} \text{ and } \mu^k(c) = \frac{0.5\sigma_1^k(y)}{0.5\sigma_1^k(y) + 0.5\sigma_1^k(y)} = \frac{1}{2}.$$



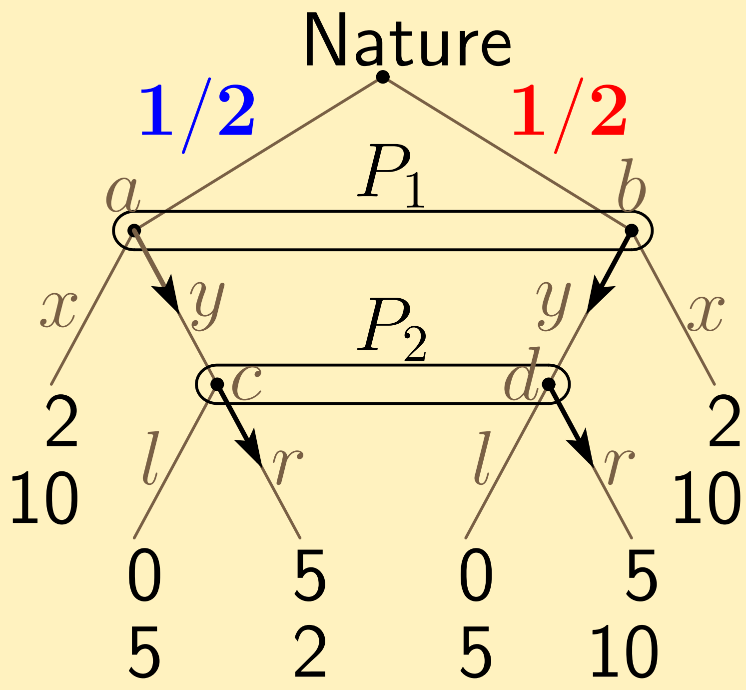

$\lim_{k\to\infty} \mu^k(a) = 1/2$,
$\lim_{k\to\infty} \mu^k(c) = 1/2$.

$$E(v_2|h_2, \mu, \mathbf{r}, \sigma_1) = (1/2)2 + (1/2)10$$
$$> E(v_2|h_2, \mu, \mathbf{l}, \sigma_1) = (1/2)5 + (1/2)5.$$
$$\Rightarrow \sigma_2(r) = 1,\ \sigma_2(l) = 0.$$

## Ex. 9.C.4 Reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{1}{2} \text{ and } \mu^k(c) = \frac{0.5\sigma_1^k(y)}{0.5\sigma_1^k(y) + 0.5\sigma_1^k(y)} = \frac{1}{2}.$$




$\lim_{k\to\infty} \mu^k(a) = 1/2,$
$\lim_{k\to\infty} \mu^k(c) = 1/2.$
$\Rightarrow \sigma_2(r) = 1,\ \sigma_2(l) = 0.$

$$E(v_1|h_1, \mu, \mathbf{y}, \sigma_2) = 5$$
$$> E(v_1|h_1, \mu, \mathbf{x}, \sigma_2) = 2.$$

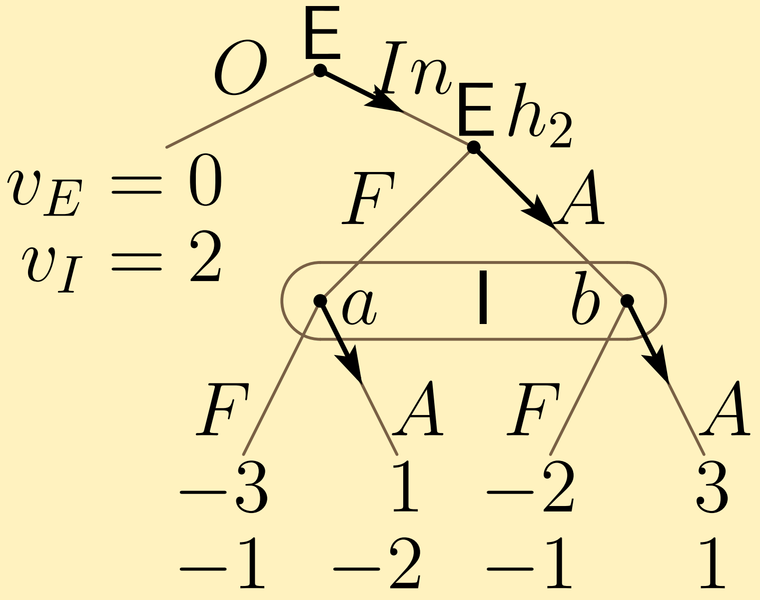
$\Rightarrow \sigma_1(y) = 1,\ \sigma_1(x) = 0.$

**Ex. 9.C.5 reconsidered**

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$

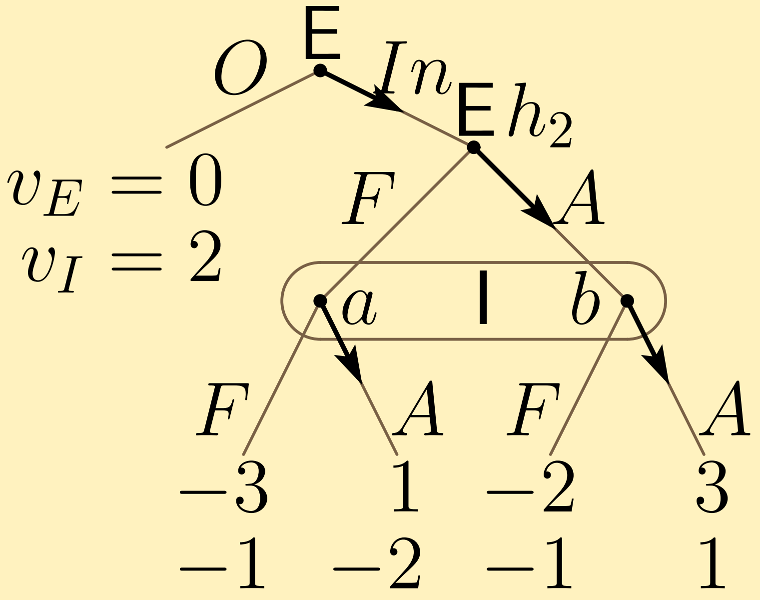


## Ex. 9.C.5 reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$



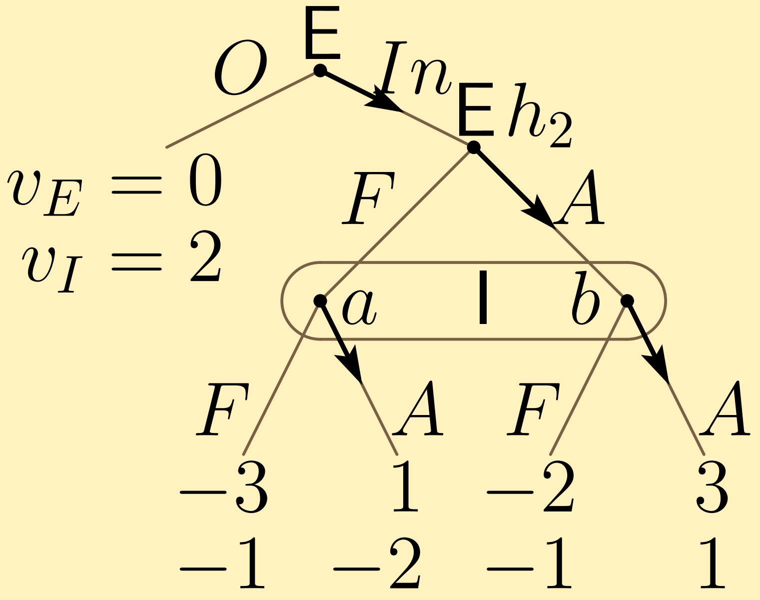

$$E(v_E|h_2, \mu, \sigma_E(In), \mathbf{F}, \sigma_I) < E(v_E|h_2, \mu, \sigma_E(In), \mathbf{A}, \sigma_I).$$

**Ex. 9.C.5 reconsidered**

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$




$$E(v_E|h_2, \mu, \sigma_E(In), \mathbf{F}, \sigma_I) < E(v_E|h_2, \mu, \sigma_E(In), \mathbf{A}, \sigma_I).$$

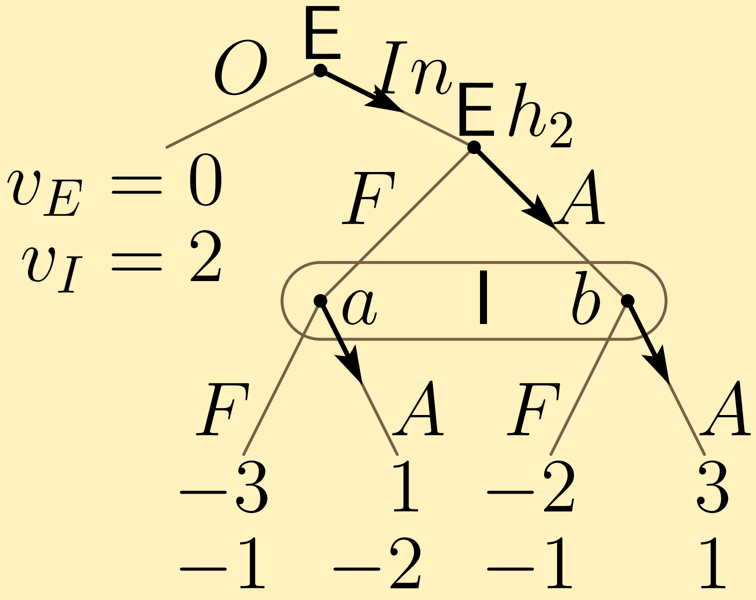
$\Rightarrow \sigma_E(F) = 0,\ \sigma_E(A) = 1.$

**Ex. 9.C.5 reconsidered**

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$




$\sigma_E(F) = 0$, $\sigma_E(A) = 1$.

$E(v_I|h_I, \mu, \sigma_E, \mathbf{F}) < E(v_I|h_I, \mu, \sigma_E, \mathbf{A})$.

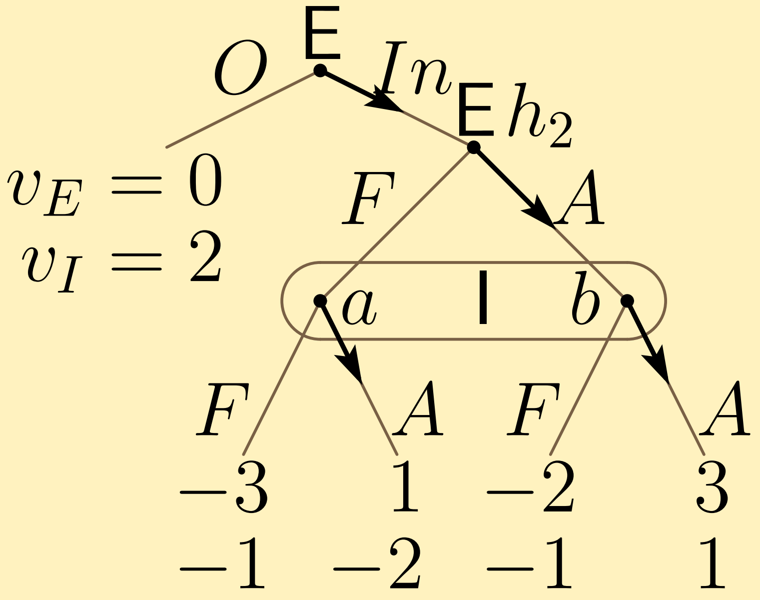
# Example

**Ex. 9.C.5 reconsidered**

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$




$\sigma_E(F) = 0$, $\sigma_E(A) = 1$.

$E(v_I|h_I, \mu, \sigma_E, \mathbf{F}) < E(v_I|h_I, \mu, \sigma_E, \mathbf{A})$.

$\Rightarrow \sigma_I(F) = 0$, $\sigma_I(A) = 1$.

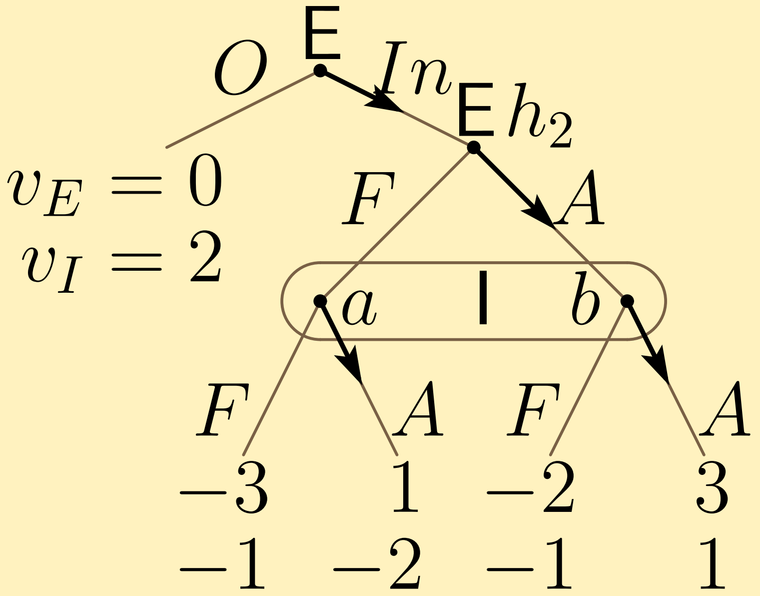
# Example

## Ex. 9.C.5 reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$




$\sigma_E(F) = 0$, $\sigma_E(A) = 1$.

$\sigma_I(F) = 0$, $\sigma_I(A) = 1$.

$E(u_E|H_1, \mu, \sigma_I, \mathbf{Out})$

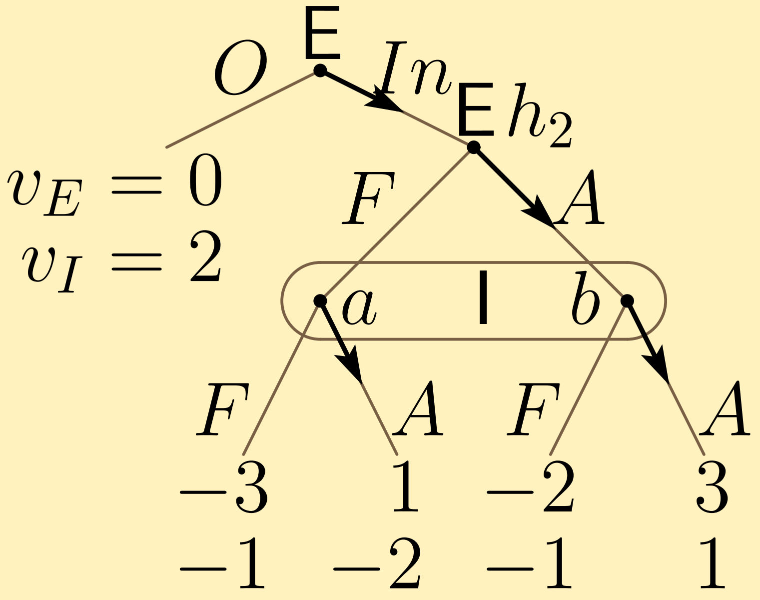
$< E(u_E|H_1, \mu, \sigma_I, \mathbf{In})$.

## Ex. 9.C.5 reconsidered

Let $(\sigma, \mu)$ be a sequential equilibrium. Let $\sigma^k \to \sigma$.

$$\forall k,\ \mu^k(a) = \frac{\Pr(a|\sigma^k)}{\Pr(h_I|\sigma^k)} = \frac{\sigma_E^k(In) \times \sigma_E^k(F)}{\sigma_E^k(In)} = \sigma_E^k(F).$$

$$\mu(a) = \lim_{k\to\infty} \mu^k(a) = \lim_{k\to\infty} \sigma_E^k(F) = \sigma_E(F).$$



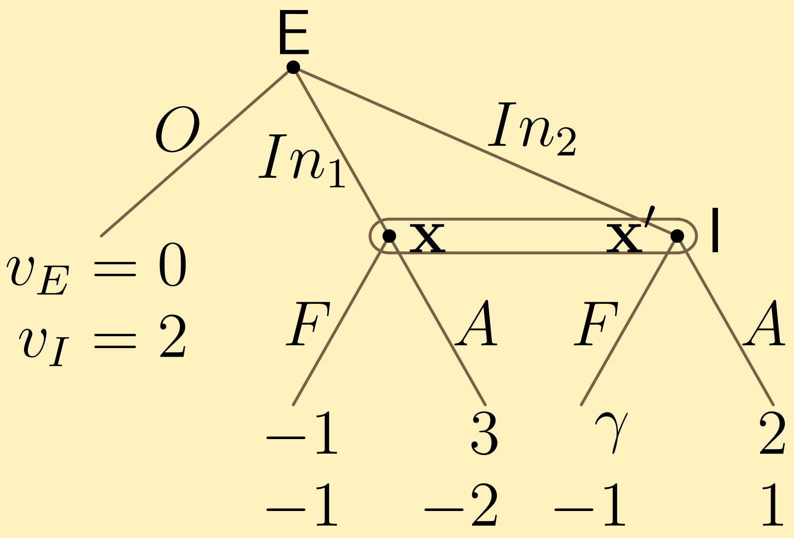

$\sigma_E(F) = 0,\ \sigma_E(A) = 1.$

$\sigma_I(F) = 0,\ \sigma_I(A) = 1.$

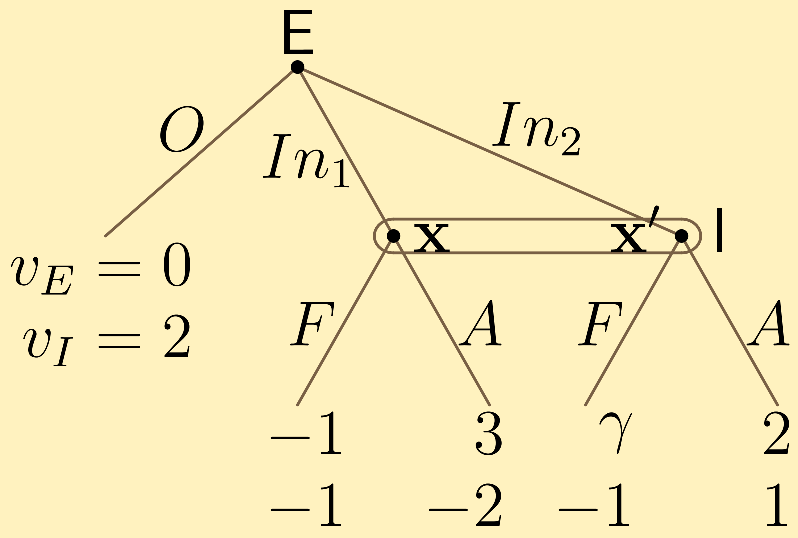
$E(u_E|H_1, \mu, \sigma_I, \mathbf{Out})$
. $< E(u_E|H_1, \mu, \sigma_I, \mathbf{In}).$

$((\sigma_E, \sigma_I), (\mu(a), \mu(b)))$
. $= (((In, A), A), (0, 1)).$

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$



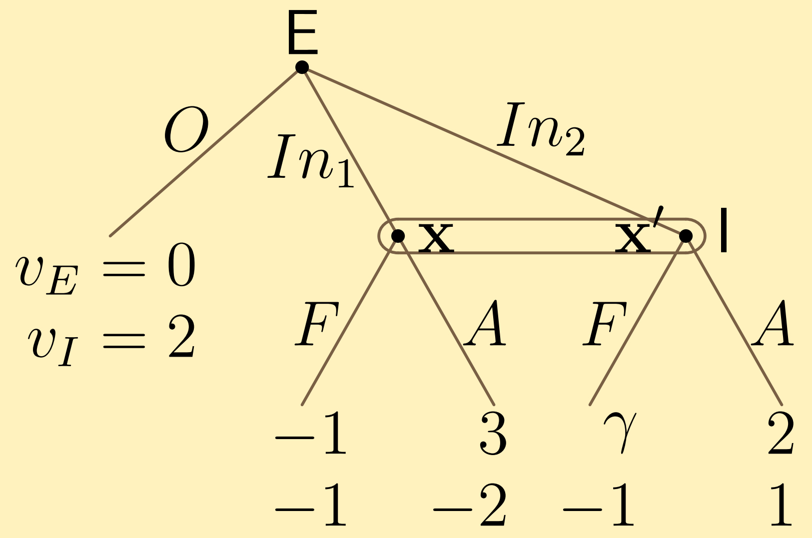
**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$




$$E[v_I(F, \sigma_E)|\mu] \gtreqless E[v_I(A, \sigma_E)|\mu] \Leftrightarrow \mu(\mathbf{x}) \gtreqless 2/3.$$

# Example

**Example 9.C.3** $(\gamma > -1,\, \gamma \neq 0)$



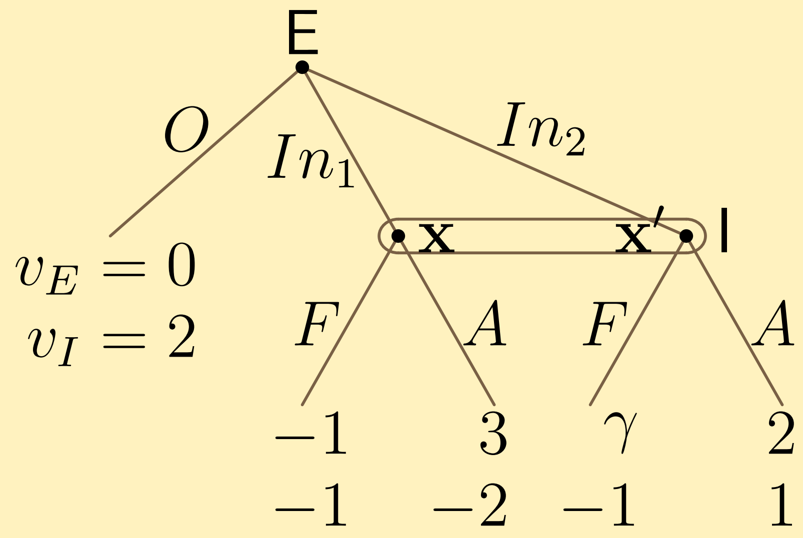

$$E[v_I(F, \sigma_E)|\mu] \gtreqless E[v_I(A, \sigma_E)|\mu] \Leftrightarrow \mu(\mathbf{x}) \gtreqless 2/3.$$

$(\gamma > 0)$:

$\mu(\mathbf{x}) > 2/3 \Rightarrow \sigma_I(F) = 1 \Rightarrow \sigma_E(In_2) = 1 \Rightarrow \mu(\mathbf{x}) = 0.$

# Example

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$



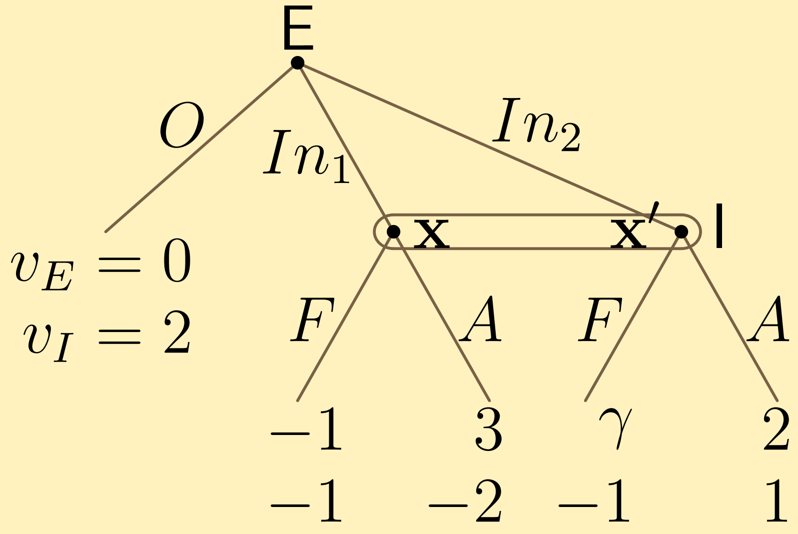

$$E[v_I(F,\sigma_E)|\mu] \gtreqless E[v_I(A,\sigma_E)|\mu] \Leftrightarrow \mu(\mathbf{x}) \gtreqless 2/3.$$

$(\gamma < 0)$:

$\mu(\mathbf{x}) > 2/3 \Rightarrow \sigma_I(F) = 1 \Rightarrow \sigma_E(In_1) = \sigma_E(In_2) = 0.$

# Example

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$




$E[v_I(F, \sigma_E)|\mu] \gtreqless E[v_I(A, \sigma_E)|\mu] \Leftrightarrow \mu(\mathbf{x}) \gtreqless 2/3.$

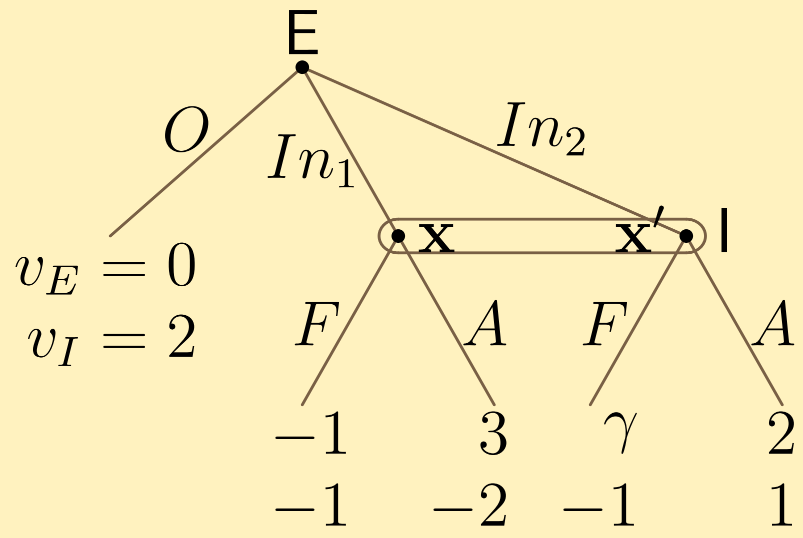
$(\gamma < 0)$:

$\mu(\mathbf{x}) > 2/3 \Rightarrow \sigma_I(F) = 1 \Rightarrow \sigma_E(In_1) = \sigma_E(In_2) = 0.$

$*$ If $\gamma < 0$, $\sigma_E(O) = 1$ and $\sigma_I(F) = 1$ with $\mu(\mathbf{x}) \geq 2/3$.

# Example

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$



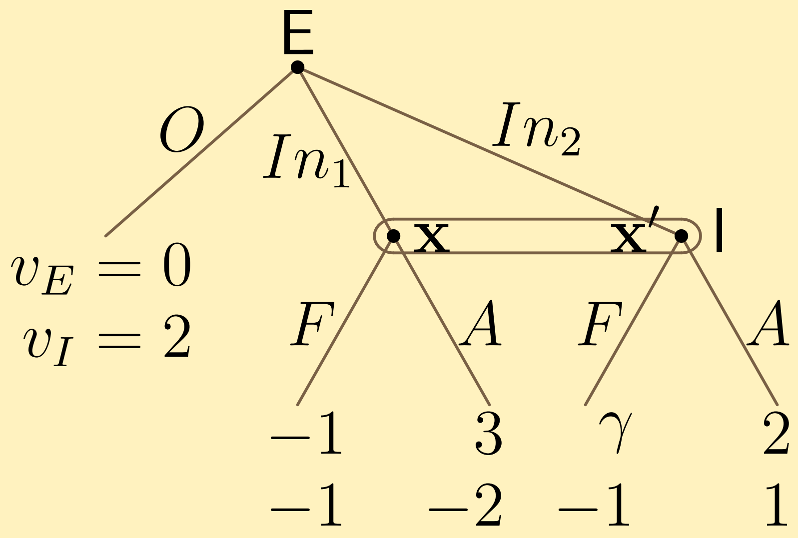

$E[v_I(F, \sigma_E)|\mu] \gtreqless E[v_I(A, \sigma_E)|\mu] \Leftrightarrow \mu(\mathbf{x}) \gtreqless 2/3.$

$*$ If $\gamma < 0$, $\sigma_E(O) = 1$ and $\sigma_I(F) = 1$ with $\mu(\mathbf{x}) \geq 2/3$.

$\mu(\mathbf{x}) < 2/3 \Rightarrow \sigma_I(F) = 0 \Rightarrow \sigma_E(In_1) = 1 \Rightarrow \mu(\mathbf{x}) = 1.$

# Example

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$

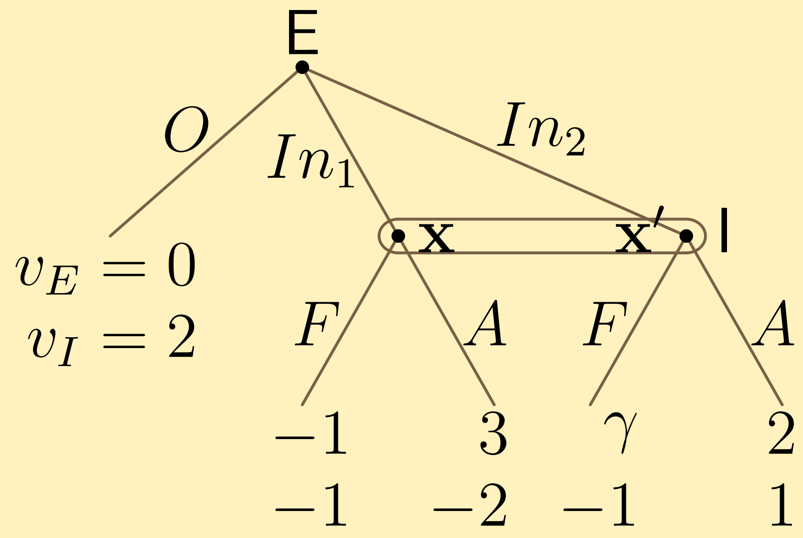



$*$ If $\gamma < 0$, $\sigma_E(O) = 1$ and $\sigma_I(F) = 1$ with $\mu(\mathbf{x}) \geq 2/3$.

$\mu(\mathbf{x}) = 2/3$, then $E[v_E(\sigma_I, In_1)|\mu] = E[v_E(\sigma_I, In_2)|\mu]$.

# Example

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$

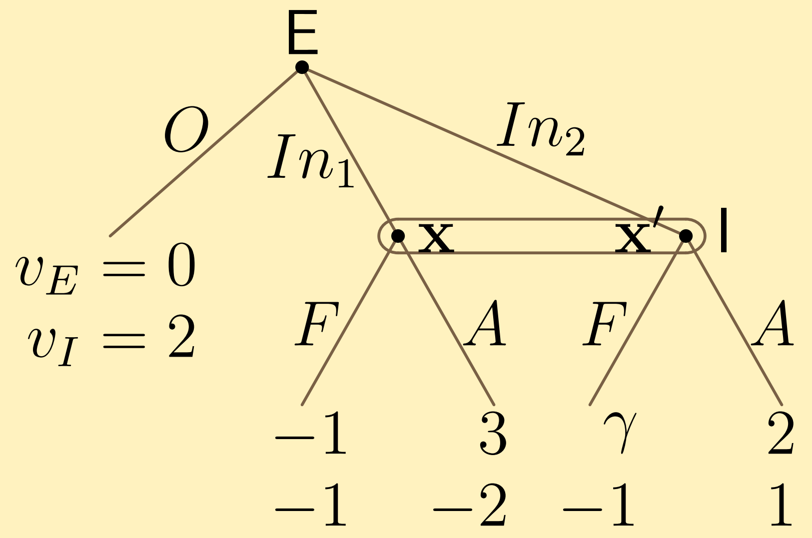



$*$ If $\gamma < 0$, $\sigma_E(O) = 1$ and $\sigma_I(F) = 1$ with $\mu(\mathbf{x}) \geq 2/3$.

$\mu(\mathbf{x}) = 2/3$, then $E[v_E(\sigma_I, In_1)|\mu] = E[v_E(\sigma_I, In_2)|\mu]$.

$\sigma_I(F) \times (-1) + (1 - \sigma_I(F)) \times 3 = \sigma_I(F) \times \gamma + (1 - \sigma_I(F)) \times 2.$

**Example 9.C.3** $(\gamma > -1, \gamma \neq 0)$

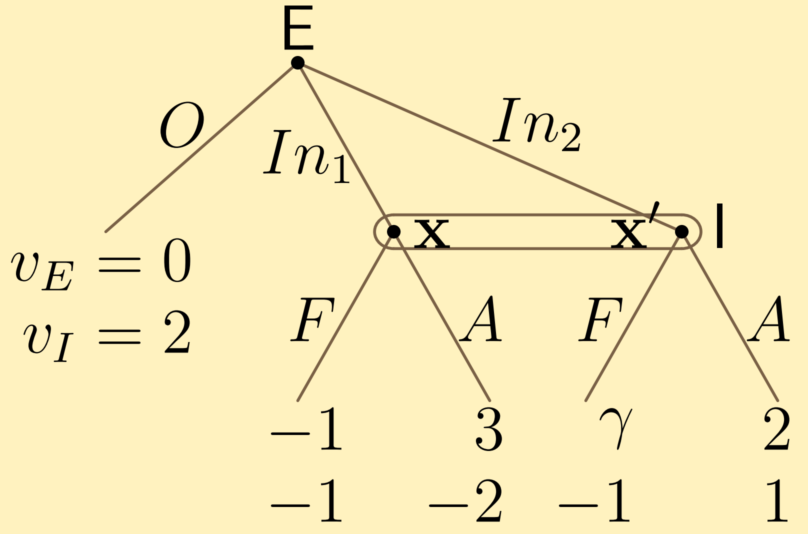



$*$ If $\gamma < 0$, $\sigma_E(O) = 1$ and $\sigma_I(F) = 1$ with $\mu(\mathbf{x}) \geq 2/3$.

$\sigma_I(F) \times (-1) + (1 - \sigma_I(F)) \times 3 = \sigma_I(F) \times \gamma + (1 - \sigma_I(F)) \times 2.$

Solving it wrt $\sigma_I(F)$, we obtain
$\sigma_I(F) = 1/(\gamma + 2)$, $E[v_E(\sigma_I, In_1)|\mu] = (3\gamma + 2)/(\gamma + 2)$.

**Example 9.C.3** $(\gamma > -1,\ \gamma \neq 0)$




$*$ If $\gamma < 0$, $\sigma_E(O) = 1$ and $\sigma_I(F) = 1$ with $\mu(\mathbf{x}) \geq 2/3$.

$$\sigma_I(F)\times(-1)+(1-\sigma_I(F))\times 3 = \sigma_I(F)\times\gamma+(1-\sigma_I(F))\times 2.$$

$$\sigma_I(F) = 1/(\gamma+2),\ \ E[v_E(\sigma_I, In_1)|\mu] = (3\gamma+2)/(\gamma+2).$$

$*$ If $\gamma \geq -2/3$, $\sigma_E(O) = 0$, $\sigma_E(In_1) = 2/3$, $\sigma_E(In_2) = 1/3$.