Technical Appendix

In this appendix, we provide the detailed proof of the results in our paper.We have used Mathematica for the calculations and figures included in this appendix.

Proof of Proposition 1

1. Second period

We start with the second-period prices.

We define four prices:
p_{Ao} is the price of Firm A for its old customers,
p_{An} is the price of Firm A for its new customers,
p_{Bo} is the price of Firm B for its old customers,
p_{Bn} is the price of Firm B for its new customers.

The location of the indifferent consumer in Firm A's turf, z_A, is derived by solving the following equation with respect to z_A:

```
Solve[-pAo - t (zA - a)^2 == -pBn - t (zA - b)^2, zA]
```

$$\left\{\left\{z_A \to \frac{a^2\, t - b^2\, t + p_{Ao} - p_{Bn}}{2\,(a-b)\,t}\right\}\right\}$$

We set the location of the indifferent consumer in Firm A's turf, z_A:

$$z_A = \frac{a^2\, t - b^2\, t + p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

$$\frac{a^2\, t - b^2\, t + p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

The location of the indifferent consumer in Firm B's turf, z_B, is derived by solving the following equation with respect to z_B:

```
Solve[-pBo - t (zB - b)^2 == -pAn - t (zB - a)^2, zB]
```

$$\left\{\left\{z_B \to \frac{a^2\, t - b^2\, t + p_{An} - p_{Bo}}{2\,(a-b)\,t}\right\}\right\}$$

We set the location of the indifferent consumer in Firm B's turf, z_B:

$$z_B = \frac{a^2\, t - b^2\, t + p_{An} - p_{Bo}}{2\,(a-b)\,t}$$

$$\frac{a^2\, t - b^2\, t + p_{An} - p_{Bo}}{2\,(a-b)\,t}$$

The first-order differentials of Firm A's profit with respect to p_{Ao} and p_{An} are

```
Factor[D[pAo zA + pAn (zB - z), pAo]]
```

$$\frac{a^2\, t - b^2\, t + 2\, p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

```
Factor[D[pAo zA + pAn (zB - z), pAn]]
```

$$-\frac{-a^2\, t + b^2\, t + 2\, a\, t\, z - 2\, b\, t\, z - 2\, p_{An} + p_{Bo}}{2\,(a-b)\,t}$$

Similarly, the first-order differentials of Firm B's profit with respect to p_{Bo} and p_{Bn} are

```
Factor[D[p_Bn (z - z_A) + p_Bo (1 - z_B), p_Bo]]
```

$$-\frac{-2\,a\,t + a^2\,t + 2\,b\,t - b^2\,t + p_{An} - 2\,p_{Bo}}{2\,(a-b)\,t}$$

```
Factor[D[p_Bn (z - z_A) + p_Bo (1 - z_B), p_Bn]]
```

$$\frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - p_{Ao} + 2\,p_{Bn}}{2\,(a-b)\,t}$$

The four first-order differentials give us the following simultaneous equations, which we solve for the second-period prices.

$$\text{Simplify}\Big[\text{Solve}\Big[\Big\{\frac{a^2\,t - b^2\,t + 2\,p_{Ao} - p_{Bn}}{2\,(a-b)\,t} == 0,\; -\frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - 2\,p_{An} + p_{Bo}}{2\,(a-b)\,t} == 0,\; -\frac{-2\,a\,t + a^2\,t + 2\,b\,t - b^2\,t + p_{An} - 2\,p_{Bo}}{2\,(a-b)\,t} == 0,\; \frac{-a^2\,t + b^2\,t + 2\,a\,t\,z - 2\,b\,t\,z - p_{Ao} + 2\,p_{Bn}}{2\,(a-b)\,t} == 0\Big\},\; \{p_{Ao}, p_{An}, p_{Bn}, p_{Bo}\}\Big]\Big]$$

$$\Big\{\Big\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\; p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),\; p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\; p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}\Big\}$$

Substituting the above prices into z_A, we have the equilibrium z_A in period 2:

$$\text{Factor}\Big[z_A\; /.\; \Big\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\; p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),\; p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\; p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}\Big]$$

$$\frac{1}{6}\,(a+b+2\,z)$$

This z_A is in the range [0, z] if and only if z ≥ (a + b)/4. If z≤(a+b)/4, we need to consider a corner solution (z_A=z), which is discussed later.

Similarly, substituting the above prices into z_B, we have the equilibrium z_B in period 2:

$$\text{Factor}\Big[z_B\; /.\; \Big\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2\,z),\; p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4\,z),\; p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4\,z),\; p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2\,z)\Big\}\Big]$$

$$\frac{1}{6}\,(2+a+b+2\,z)$$

This z_B is in the range [z, 1] if and only if z ≤ (2+a+b)/4. If z≥(2+a+b)/4, we need to consider a corner solution (z_B=z), which is discussed later.

Based on the above discussions, we have three cases: (i) 0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i) 0≤z≤ (a + b)/4.

In this case, we have $z_A = z$, hence Firm B cannot poach any customer in Firm A's turf. As a result, p_{Bn}=0. Anticipating this, Firm A sets the highest p_{Ao} that leads to $z_A = z$. z_A, just equals to z. This is found below.

```
Solve[{z_A == z, p_Bn == 0}, {p_Ao, p_Bn}]
```

$$\{\{p_{Ao} \to -(a-b)\,t\,(a+b-2\,z),\; p_{Bn} \to 0\}\}$$

For the optimal pricing in Firm B's turf, we can use the first-order differentials we have already derived:

The first - order differential of Firm A (p_{An}) : $-\frac{-a^2 t + b^2 t + 2 a t z - 2 b t z - 2 p_{An} + p_{Bo}}{2 (a - b) t}$

The first - order differential of Firm B (p_{Bo}) : $-\frac{-2 a t + a^2 t + 2 b t - b^2 t + p_{An} - 2 p_{Bo}}{2 (a - b) t}$

This leads to the following second-period prices:

$$\text{Simplify}\left[\text{Solve}\left[\left\{-\frac{-a^2 t + b^2 t + 2 a t z - 2 b t z - 2 p_{An} + p_{Bo}}{2 (a - b) t} == 0, -\frac{-2 a t + a^2 t + 2 b t - b^2 t + p_{An} - 2 p_{Bo}}{2 (a - b) t} == 0\right\}, \{p_{An}, p_{Bo}\}\right]\right]$$

$$\left\{\left\{p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z)\right\}\right\}$$

Substituting the above prices into z_B, we have the equilibrium z_B in period 2 when 0≤z≤(a+b)/4:

$$\text{Factor}\left[z_B \text{ /. } \left\{p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z)\right\}\right]$$

$$\frac{1}{6} (2 + a + b + 2 z)$$

Using the above outcomes, we derive the second period profit of Firm A in case (i) given the first-period z:

$$\text{Factor}\left[p_{Ao} z_A + p_{An} (z_B - z) \text{ /. } \left\{p_{Ao} \to -(a - b) t (a + b - 2 z), p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z), p_{Bn} \to 0\right\}\right]$$

$$-\frac{1}{18} (a - b) t \left(4 + 4 a + a^2 + 4 b + 2 a b + b^2 - 16 z + 10 a z + 10 b z - 20 z^2\right)$$

Similarly, using the above outcomes, we derive the second period profit of Firm B in case (i) given the first-period z:s

$$\text{Factor}\left[p_{Bn} (z - z_A) + p_{Bo} (1 - z_B) \text{ /. } \left\{p_{Ao} \to -(a - b) t (a + b - 2 z), p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z), p_{Bn} \to 0\right\}\right]$$

$$-\frac{1}{18} (a - b) t (-4 + a + b + 2 z)^2$$

(Case ii) (a+b)/4<z<(2+a+b)/4.

In this case, we have an interior solution with two-way poaching. Therefore, we can use the second-period prices we have already obtained previously, reproduced below:

$$\left\{p_{Ao} \to -\frac{1}{3} (a - b) t (a + b + 2 z), p_{An} \to -\frac{1}{3} (a - b) t (2 + a + b - 4 z), p_{Bn} \to \frac{1}{3} (a - b) t (a + b - 4 z), p_{Bo} \to \frac{1}{3} (a - b) t (-4 + a + b + 2 z)\right\}$$

Thus firm A's second-period profit given *z* can be derived as

$$\text{Factor}\left[p_{Ao}\, z_A + p_{An}\, (z_B - z)\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2z)\right\}\right]$$

$$-\frac{1}{9}\,(a-b)\,t\,\left(2+2a+a^2+2b+2ab+b^2-8z-2az-2bz+10z^2\right)$$

Similarly, firm B’s profit given z is

$$\text{Factor}\left[p_{Bn}\,(z - z_A) + p_{Bo}\,(1 - z_B)\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2z),\ p_{An} \to -\frac{1}{3}\,(a-b)\,t\,(2+a+b-4z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4z),\ p_{Bo} \to \frac{1}{3}\,(a-b)\,t\,(-4+a+b+2z)\right\}\right]$$

$$-\frac{1}{9}\,(a-b)\,t\,\left(8-4a+a^2-4b+2ab+b^2-8z-2az-2bz+10z^2\right)$$

(Case iii) (2 + a + b)/4≤z≤1.

In this case, $z_B = z$, hence Firm A cannot poach any customer in Firm B’s turf. As a result, p_{An}=0. Given p_{An}=0, Firm B chooses the highest p_{Bo} that leads to $z_B = z$:

$$\text{Solve}[\{z_B == z,\ p_{An} == 0\},\ \{p_{Bo},\ p_{An}\}]$$

$$\{\{p_{Bo} \to (a-b)\,t\,(a+b-2z),\ p_{An} \to 0\}\}$$

For the optimal pricing in Firm A’s turf, we can use the first-order differentials we have already derived:

$$\text{The first - order differential of Firm A }(p_{Ao}):\ \frac{a^2 t - b^2 t + 2 p_{Ao} - p_{Bn}}{2\,(a-b)\,t}$$

$$\text{The first - order differential of Firm B }(p_{Bn}):\ \frac{-a^2 t + b^2 t + 2atz - 2btz - p_{Ao} + 2 p_{Bn}}{2\,(a-b)\,t}$$

Solving the following simultaneous equations gives us the second-period prices.

$$\text{Simplify}\left[\text{Solve}\left[\left\{\frac{a^2 t - b^2 t + 2 p_{Ao} - p_{Bn}}{2\,(a-b)\,t} == 0,\ \frac{-a^2 t + b^2 t + 2atz - 2btz - p_{Ao} + 2 p_{Bn}}{2\,(a-b)\,t} == 0\right\},\ \{p_{Ao},\ p_{Bn}\}\right]\right]$$

$$\left\{\left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4z)\right\}\right\}$$

Substituting the second-period prices into z_A, we have the equilibrium z_A in period 2 when z ≥ (2+a + b)/4:

$$\text{Factor}\left[z_A\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2z),\ p_{An} \to 0,\ p_{Bo} \to (a-b)\,t\,(a+b-2z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4z)\right\}\right]$$

$$\frac{1}{6}\,(a+b+2z)$$

Using the above prices, z_A and $z_B = z$, we derive firm A’s second-period profit given z as follows:

$$\text{Factor}\left[p_{Ao}\, z_A + p_{An}\,(z_B - z)\ /.\ \left\{p_{Ao} \to -\frac{1}{3}\,(a-b)\,t\,(a+b+2z),\ p_{An} \to 0,\ p_{Bo} \to (a-b)\,t\,(a+b-2z),\ p_{Bn} \to \frac{1}{3}\,(a-b)\,t\,(a+b-4z)\right\}\right]$$

$$-\frac{1}{18}\,(a-b)\,t\,(a+b+2z)^2$$

Similarly, firm B’s second-period profit given z is

Factor[p_{Bn} (z - z_A) + p_{Bo} (1 - z_B) /.
$\{p_{Ao} \to -\frac{1}{3}$ (a - b) t (a + b + 2 z), $p_{An} \to 0$, $p_{Bo} \to$ (a - b) t (a + b - 2 z), $p_{Bn} \to \frac{1}{3}$ (a - b) t (a + b - 4 z)$\}$]

$$-\frac{1}{18} (a-b)\, t \left(-18 a + a^2 - 18 b + 2 a b + b^2 + 36 z + 10 a z + 10 b z - 20 z^2\right)$$

2. First period - Prices

In our calculation, we denote firms' discount factor by δ_f and consumers' discount factor by δ_c. In Proposition 1, we focus on the case, δ_f=δ_f=δ. In Propositions 2 and 3, we focus on the case, δ_f=δ and δ_c=0.
We need to consider three cases: (i) 0≤z ≤ (a + b)/4, (ii) (a + b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤ z≤1.

(Case i) 0≤z≤(a + b)/4.

From the previous analysis, we have the second-period prices given as follow.

$$\left\{p_{Ao} \to -(a-b)\, t\, (a+b-2z),\ p_{An} \to -\frac{1}{3} (a-b)\, t\, (2+a+b-4z),\ p_{Bo} \to \frac{1}{3} (a-b)\, t\, (-4+a+b+2z),\ p_{Bn} \to 0\right\}$$

Anticipating the second period prices, consumers choose one of the first-period prices p_A or p_B (p_A is the first-period price of firm A and p_B is the first-period price of firm B)

The location of the indifferent consumer, *z*, is derived from the following equation.:

Solve[$-p_A - t (z-a)^2 - \delta c\, \left((-(a-b)\, t\, (a+b-2z)) + t (z-a)^2\right) ==$
$-p_B - t (z-b)^2 - \delta c \left(\left(\frac{1}{3} (b-a)\, t\, (2+a+b-4z)\right) + t (z-a)^2\right)$, z]

$$\left\{\left\{z \to \left(-3 a^2 t + 3 b^2 t - 2 a t \delta c + 2 a^2 t \delta c + 2 b t \delta c - 2 b^2 t \delta c - 3 p_A + 3 p_B\right) / (2 (a-b)\, t\, (-3+\delta c))\right\}\right\}$$

We set the location of the indifferent consumers *z*:

z = $\left(-3 a^2 t + 3 b^2 t - 2 a t \delta c + 2 a^2 t \delta c + 2 b t \delta c - 2 b^2 t \delta c - 3 p_A + 3 p_B\right) / (2 (a-b)\, t\, (-3+\delta c))$

$$\frac{-3 a^2 t + 3 b^2 t - 2 a t \delta c + 2 a^2 t \delta c + 2 b t \delta c - 2 b^2 t \delta c - 3 p_A + 3 p_B}{2 (a-b)\, t\, (-3+\delta c)}$$

Next, we derive the condition for *z* to be in the range [0, (a+b)/4] by solving the following equations.

Factor[Solve[(a + b) / 4 - z == 0, p_A]]
Factor[Solve[0 - z == 0, p_A]]

$$\left\{\left\{p_A \to \frac{1}{6} \left(-3 a^2 t + 3 b^2 t - 4 a t \delta c + 3 a^2 t \delta c + 4 b t \delta c - 3 b^2 t \delta c + 6 p_B\right)\right\}\right\}$$

$$\left\{\left\{p_A \to \frac{1}{3} \left(-3 a^2 t + 3 b^2 t - 2 a t \delta c + 2 a^2 t \delta c + 2 b t \delta c - 2 b^2 t \delta c + 3 p_B\right)\right\}\right\}$$

Simplifying the above, the condition can be stated as follows.

$$p_B + \frac{(b-a)\, t\, ((a+b)\, (3-2\delta c) + 2\delta c)}{3} \geq p_A \geq p_B + \frac{(b-a)\, t\, (3 (a+b)\, (1-\delta c) + 4\delta c)}{6}$$

Note that if p_A is larger than $p_B + \frac{(b-a)\, t\, ((a+b)\, (3-2\delta c) + 2\delta c)}{3}$, *z* becomes zero.

Next, we solve for the pricing equilibrium in the first period.
From the previous analysis, we have each firm’s second-period profit given as follows.

π_{A2} : $-\frac{1}{18}$ (a − b) t $\left(4 + 4 a + a^2 + 4 b + 2 a b + b^2 - 16 z + 10 a z + 10 b z - 20 z^2\right)$

π_{B2} : $-\frac{1}{18}$ (a − b) t $(-4 + a + b + 2 z)^2$

First, the derivative of firm A's total discounted profit with respect to p_A is

$$\text{Factor}\left[D\left[p_A z + \delta f \left(\frac{1}{18} (b - a)\, t \left(4 + 4 a + a^2 + 4 b + 2 a b + b^2 - 16 z + 10 a z + 10 b z - 20 z^2\right)\right),\ p_A\right]\right]$$

$$\frac{1}{2 (a - b)\, t\, (-3 + \delta c)^2} \left(9 a^2 t - 9 b^2 t + 6 a t \delta c - 9 a^2 t \delta c - 6 b t \delta c + 9 b^2 t \delta c - 2 a t \delta c^2 + 2 a^2 t \delta c^2 + 2 b t \delta c^2 - 2 b^2 t \delta c^2 + 8 a t \delta f + 5 a^2 t \delta f - 8 b t \delta f - 5 b^2 t \delta f + 4 a t \delta c\, \delta f - 5 a^2 t \delta c\, \delta f - 4 b t \delta c\, \delta f + 5 b^2 t \delta c\, \delta f + 18 p_A - 6 \delta c\, p_A + 10 \delta f\, p_A - 9 p_B + 3 \delta c\, p_B - 10 \delta f\, p_B\right)$$

Using the derivative, we obtain the reaction function of Firm A in the range, 0≤z≤(a + b)/4:

$$\text{Simplify}\Big[\text{Solve}\Big[\frac{1}{2 (a - b)\, t\, (-3 + \delta c)^2} \left(9 a^2 t - 9 b^2 t + 6 a t \delta c - 9 a^2 t \delta c - 6 b t \delta c + 9 b^2 t \delta c - 2 a t \delta c^2 + 2 a^2 t \delta c^2 + 2 b t \delta c^2 - 2 b^2 t \delta c^2 + 8 a t \delta f + 5 a^2 t \delta f - 8 b t \delta f - 5 b^2 t \delta f + 4 a t \delta c\, \delta f - 5 a^2 t \delta c\, \delta f - 4 b t \delta c\, \delta f + 5 b^2 t \delta c\, \delta f + 18 p_A - 6 \delta c\, p_A + 10 \delta f\, p_A - 9 p_B + 3 \delta c\, p_B - 10 \delta f\, p_B\right) == 0,\ p_A\Big]\Big]$$

$$\left\{\left\{p_A \to \frac{1}{2 (-9 + 3 \delta c - 5 \delta f)} \left((a - b)\, t \left(6 \delta c - 2 \delta c^2 + 8 \delta f + 4 \delta c\, \delta f + b \left(9 - 9 \delta c + 2 \delta c^2 + 5 \delta f - 5 \delta c\, \delta f\right) + a \left(9 + 2 \delta c^2 + 5 \delta f - \delta c\, (9 + 5 \delta f)\right)\right) + (-9 + 3 \delta c - 10 \delta f)\, p_B\right)\right\}\right\}$$

Since the above reaction function may prescribe *z* outside the required range, we need the condition that indeed guarantees,0≤z≤(a + b)/4.

We have already obtained the condition that z is between 0 and (a + b)/4 as follows:

$$p_B + \frac{(b - a)\, t\, ((a + b)\, (3 - 2 \delta c) + 2 \delta c)}{3} \geq p_A \geq p_B + \frac{(b - a)\, t\, (3 (a + b)\, (1 - \delta c) + 4 \delta c)}{6}$$

If the following outcomes are positive, the reaction function satisfies the above inequalities:

$$\text{Simplify}\Big[\text{Factor}\Big[\frac{1}{2 (-9 + 3 \delta c - 5 \delta f)} \left((a - b)\, t \left(6 \delta c - 2 \delta c^2 + 8 \delta f + 4 \delta c\, \delta f + b \left(9 - 9 \delta c + 2 \delta c^2 + 5 \delta f - 5 \delta c\, \delta f\right) + a \left(9 + 2 \delta c^2 + 5 \delta f - \delta c\, (9 + 5 \delta f)\right)\right) + (-9 + 3 \delta c - 10 \delta f)\, p_B\right) - \left(p_B + \frac{(b - a)\, t\, (3 (a + b)\, (1 - \delta c) + 4 \delta c)}{6}\right)\Big]\Big]$$

$$\text{Simplify}\Big[\text{Factor}\Big[p_B + \frac{(b - a)\, t\, ((a + b)\, (3 - 2 \delta c) + 2 \delta c)}{3} - \frac{1}{2 (-9 + 3 \delta c - 5 \delta f)} \left((a - b)\, t \left(6 \delta c - 2 \delta c^2 + 8 \delta f + 4 \delta c\, \delta f + b \left(9 - 9 \delta c + 2 \delta c^2 + 5 \delta f - 5 \delta c\, \delta f\right) + a \left(9 + 2 \delta c^2 + 5 \delta f - \delta c\, (9 + 5 \delta f)\right)\right) + (-9 + 3 \delta c - 10 \delta f)\, p_B\right)\Big]\Big]$$

$$-\frac{(-3 + \delta c)\, ((a - b)\, t\, (3 (-2 + a + b)\, \delta c + 8 \delta f) + 9 p_B)}{6 (-9 + 3 \delta c - 5 \delta f)}$$

$$((-3 + \delta c)\, ((a - b)\, t\, (-6 \delta c + a (-9 + 6 \delta c - 5 \delta f) + b (-9 + 6 \delta c - 5 \delta f) + 8 \delta f) + 9 p_B)) / (6 (-9 + 3 \delta c - 5 \delta f))$$

We derive the threshold values of p_B such that each of the outcomes equals zero:

$$\text{Simplify}\left[\text{Solve}\left[-\frac{(-3+\delta c)\ ((a-b)\ t\ (3\ (-2+a+b)\ \delta c+8\ \delta f)+9\ p_B)}{6\ (-9+3\ \delta c-5\ \delta f)}==0,\ p_B\right]\right]$$

```
Simplify[Solve[((-3 + δc) ((a - b) t (-6 δc + a (-9 + 6 δc - 5 δf) + b (-9 + 6 δc - 5 δf) + 8 δf) + 9 pB)) /
    (6 (-9 + 3 δc - 5 δf)) == 0, pB]]
```

$$\left\{\left\{p_B\to-\frac{1}{9}\ (a-b)\ t\ (3\ (-2+a+b)\ \delta c+8\ \delta f)\right\}\right\}$$

$$\left\{\left\{p_B\to-\frac{1}{9}\ (a-b)\ t\ (-6\ \delta c+a\ (-9+6\ \delta c-5\ \delta f)+b\ (-9+6\ \delta c-5\ \delta f)+8\ \delta f)\right\}\right\}$$

Therefore, if p_B satisfies the following inequalities, the reaction function of Firm A is in the range, 0≤z≤(a + b)/4:

$$-\frac{1}{9}\ (b-a)\ t\ ((a+b)\ (9-6\ \delta c+5\ \delta f)+6\ \delta c-8\ \delta f)\le p_B\le\frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c+8\ \delta f)}{9}$$

Note that if p_B is smaller than the left-hand side value of the inequality, Firm A abandons to supply in period 1.

Similarly, using the above outcomes, we derive the first-order derivative of Firm B's profit with respect to p_B

$$\text{Factor}\left[D\left[p_B\ (1-z)+\delta f\ \left(\frac{1}{18}\ (b-a)\ t\ (-4+a+b+2\ z)^2\right),\ p_B\right]\right]$$

$$-\frac{1}{2\ (a-b)\ t\ (-3+\delta c)^2}\left(-18\ a\ t+9\ a^2\ t+18\ b\ t-9\ b^2\ t+18\ a\ t\ \delta c-9\ a^2\ t\ \delta c-18\ b\ t\ \delta c+9\ b^2\ t\ \delta c-4\ a\ t\ \delta c^2+2\ a^2\ t\ \delta c^2+4\ b\ t\ \delta c^2-2\ b^2\ t\ \delta c^2+8\ a\ t\ \delta f-4\ a^2\ t\ \delta f-8\ b\ t\ \delta f+4\ b^2\ t\ \delta f-4\ a\ t\ \delta c\ \delta f+2\ a^2\ t\ \delta c\ \delta f+4\ b\ t\ \delta c\ \delta f-2\ b^2\ t\ \delta c\ \delta f+9\ p_A-3\ \delta c\ p_A-2\ \delta f\ p_A-18\ p_B+6\ \delta c\ p_B+2\ \delta f\ p_B\right)$$

Using the derivative, we obtain the reaction function of Firm B in the range, 0≤z ≤ (a + b)/4:

$$\text{Simplify}\left[\text{Solve}\left[-\frac{1}{2\ (a-b)\ t\ (-3+\delta c)^2}\left(-18\ a\ t+9\ a^2\ t+18\ b\ t-9\ b^2\ t+18\ a\ t\ \delta c-9\ a^2\ t\ \delta c-18\ b\ t\ \delta c+9\ b^2\ t\ \delta c-4\ a\ t\ \delta c^2+2\ a^2\ t\ \delta c^2+4\ b\ t\ \delta c^2-2\ b^2\ t\ \delta c^2+8\ a\ t\ \delta f-4\ a^2\ t\ \delta f-8\ b\ t\ \delta f+4\ b^2\ t\ \delta f-4\ a\ t\ \delta c\ \delta f+2\ a^2\ t\ \delta c\ \delta f+4\ b\ t\ \delta c\ \delta f-2\ b^2\ t\ \delta c\ \delta f+9\ p_A-3\ \delta c\ p_A-2\ \delta f\ p_A-18\ p_B+6\ \delta c\ p_B+2\ \delta f\ p_B\right)==0,\ p_B\right]\right]$$

$$\left\{\left\{p_B\to\left(-\left(-2\ a+a^2-(-2+b)\ b\right)\ t\ \left(9+2\ \delta c^2-4\ \delta f+\delta c\ (-9+2\ \delta f)\right)+(-9+3\ \delta c+2\ \delta f)\ p_A\right)/\left(2\ (-9+3\ \delta c+\delta f)\right)\right\}\right\}$$

Since the above reaction function may prescribe z outside the required range, we need the condition that indeed guarantees 0≤z ≤ (a + b)/4.

We have already obtained the condition that z is between 0 and (a + b)/4 as follows:

$$p_B+\frac{(b-a)\ t\ ((a+b)\ (3-2\ \delta c)+2\ \delta c)}{3}\ge p_A\ge p_B+\frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c)+4\ \delta c)}{6}.$$

If p_A is larger than $p_B+\frac{(b-a)\ t\ ((a+b)\ (3-2\ \delta c)+2\ \delta c)}{3}$, z becomes zero. In this case, Firm B chooses the following p_B which just leads to z=0.

$$p_B\to\ p_A-\frac{(b-a)\ t\ ((a+b)\ (3-2\ \delta c)+2\ \delta c)}{3}.$$

If the following outcomes are positive, the reaction function satisfies the above inequalities, 0≤z ≤ (a + b)/4:

$$\text{Simplify}\left[\text{Factor}\left[\left(-\left(-2a+a^2-(-2+b)\,b\right)t\left(9+2\delta c^2-4\delta f+\delta c\,(-9+2\delta f)\right)+(-9+3\delta c+2\delta f)\,p_A\right)/\left(2\,(-9+3\delta c+\delta f)\right)+\frac{(b-a)\,t\,((a+b)\,(3-2\delta c)+2\delta c)}{3}-p_A\right]\right]$$

$$\text{Simplify}\left[\text{Factor}\left[p_A-\left(\left(-\left(-2a+a^2-(-2+b)\,b\right)t\left(9+2\delta c^2-4\delta f+\delta c\,(-9+2\delta f)\right)+(-9+3\delta c+2\delta f)\,p_A\right)/\left(2\,(-9+3\delta c+\delta f)\right)+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)+4\delta c)}{6}\right)\right]\right]$$

$$((-3+\delta c)\,((a-b)\,t\,(-18+a\,(-9+6\delta c-2\delta f)+b\,(-9+6\delta c-2\delta f)+8\delta f)-9\,p_A))\,/\,(6\,(-9+3\delta c+\delta f))$$

$$-(((-3+\delta c)\,((a-b)\,t\,(-18+3\,b\,\delta c+3\,a\,(\delta c-\delta f)+8\delta f-3\,b\,\delta f)-9\,p_A))\,/\,(6\,(-9+3\delta c+\delta f)))$$

We derive the threshold values of p_A such that each of the outcomes equals zero:

$$\text{Simplify}[\text{Solve}[((-3+\delta c)\,((a-b)\,t\,(-18+a\,(-9+6\delta c-2\delta f)+b\,(-9+6\delta c-2\delta f)+8\delta f)-9\,p_A))\,/\,(6\,(-9+3\delta c+\delta f))==0,\ p_A]]$$
$$\text{Simplify}[\text{Solve}[-(((-3+\delta c)\,((a-b)\,t\,(-18+3\,b\,\delta c+3\,a\,(\delta c-\delta f)+8\delta f-3\,b\,\delta f)-9\,p_A))\,/\,(6\,(-9+3\delta c+\delta f)))==0,\ p_A]]$$

$$\left\{\left\{p_A\to\frac{1}{9}\,(a-b)\,t\,(-18+a\,(-9+6\delta c-2\delta f)+b\,(-9+6\delta c-2\delta f)+8\delta f)\right\}\right\}$$

$$\left\{\left\{p_A\to\frac{1}{9}\,(a-b)\,t\,(-18+3\,a\,(\delta c-\delta f)+3\,b\,(\delta c-\delta f)+8\delta f)\right\}\right\}$$

Therefore, if p_A satisfies the following inequalities, the reaction function of Firm B is in the range, $0 \le z \le (a+b)/4$:

$$\frac{(b-a)\,t\,(18+(a+b)\,(9-6\delta c+2\delta f)-8\delta f)}{9}\ \ge p_A\ge\ \frac{(b-a)\,t\,(18-3\,(a+b)\,(\delta c-\delta f)-8\delta f)}{9}$$

Note that if p_A is larger than the left-hand side value of the inequality, Firm B chooses the following p_B, which leads to z=0.

$$p_B\to\ p_A-\frac{(b-a)\,t\,((a+b)\,(3-2\delta c)+2\delta c)}{3}$$

(Case ii) (a + b)/4 < z < (2 + a + b)/4

From the previous analysis, we have the second-period prices given as follow.

$$\left\{p_{Ao}\to-\frac{1}{3}\,(a-b)\,t\,(a+b+2z),\ p_{An}\to-\frac{1}{3}\,(a-b)\,t\,(2+a+b-4z),\ p_{Bn}\to\frac{1}{3}\,(a-b)\,t\,(a+b-4z),\ p_{Bo}\to\frac{1}{3}\,(a-b)\,t\,(-4+a+b+2z)\right\}$$

Anticipating the second period prices, consumers choose one of the first-period prices p_A or p_B (p_A is the first-period price of firm A and p_B is the first-period price of firm B)

The location of the indifferent consumer, *z*, is derived by the following equation:

$$\text{Solve}\left[-p_A-t\,(z-a)^2-\delta c\left(\frac{1}{3}\,(a-b)\,t\,(a+b-4z)+t\,(z-b)^2\right)==-p_B-t\,(z-b)^2-\delta c\left(\left(\frac{1}{3}\,(b-a)\,t\,(2+a+b-4z)\right)+t\,(z-a)^2\right),\ z\right]$$

$$\left\{\left\{z\to\frac{3\,a^2\,t-3\,b^2\,t+2\,a\,t\,\delta c-a^2\,t\,\delta c-2\,b\,t\,\delta c+b^2\,t\,\delta c+3\,p_A-3\,p_B}{2\,(a-b)\,t\,(3+\delta c)}\right\}\right\}$$

We set the location of the indifferent consumers z:

$$z = \frac{3\ (p_B - p_A)}{2\ (b - a)\ t\ (3 + \delta c)} + \frac{((a + b)\ (3 - \delta c) + 2\ \delta c)}{2\ (3 + \delta c)}$$

$$\frac{(a + b)\ (3 - \delta c) + 2\ \delta c}{2\ (3 + \delta c)} + \frac{3\ (-p_A + p_B)}{2\ (-a + b)\ t\ (3 + \delta c)}$$

We derive the condition that the location of the indifferent consumers, *z*, is between (a+b)/4 and (2+a+b)/4, by solving the following simultaneous equations

```
Simplify[Solve[z - (a + b) / 4 == 0, pA]]
Simplify[Solve[(2 + a + b) / 4 - z == 0, pA]]
```

$$\left\{\left\{p_A \to \frac{1}{6}\ ((a - b)\ t\ (3\ a\ (-1 + \delta c) + 3\ b\ (-1 + \delta c) - 4\ \delta c) + 6\ p_B)\right\}\right\}$$

$$\left\{\left\{p_A \to \frac{1}{6}\ ((a - b)\ t\ (6 + 3\ a\ (-1 + \delta c) + 3\ b\ (-1 + \delta c) - 2\ \delta c) + 6\ p_B)\right\}\right\}$$

By simplifying the above values of p_A, we have the condition that z is between (a+b)/4 and (2+a+b)/4 as follows:

$$p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) - 2\ (3 - \delta c))}{6} < p_A < p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) + 4\ \delta c)}{6}$$

We have already derived the 2nd period profits of Firms A and B as follows:

$$\pi_{A2} : -\frac{1}{9}\ (a - b)\ t\ \left(2 + 2\ a + a^2 + 2\ b + 2\ a\ b + b^2 - 8\ z - 2\ a\ z - 2\ b\ z + 10\ z^2\right)$$

$$\pi_{B2} : -\frac{1}{9}\ (a - b)\ t\ \left(8 - 4\ a + a^2 - 4\ b + 2\ a\ b + b^2 - 8\ z - 2\ a\ z - 2\ b\ z + 10\ z^2\right)$$

Using the above outcomes, we derive the first-order derivative of Firm A's total discounted profit with respect to p_A

$$\text{Factor}\left[D\left[p_A\ z + \delta f\ \frac{1}{9}\ (b - a)\ t\ \left(2 + 2\ a + a^2 + 2\ b + 2\ a\ b + b^2 - 8\ z - 2\ a\ z - 2\ b\ z + 10\ z^2\right),\ p_A\right]\right]$$

$$-\frac{1}{2\ (a - b)\ t\ (3 + \delta c)^2}\ \left(-9\ a^2\ t + 9\ b^2\ t - 6\ a\ t\ \delta c + 6\ b\ t\ \delta c - 2\ a\ t\ \delta c^2 + a^2\ t\ \delta c^2 + 2\ b\ t\ \delta c^2 - b^2\ t\ \delta c^2 - 8\ a\ t\ \delta f + 8\ a^2\ t\ \delta f + 8\ b\ t\ \delta f - 8\ b^2\ t\ \delta f + 4\ a\ t\ \delta c\ \delta f - 4\ a^2\ t\ \delta c\ \delta f - 4\ b\ t\ \delta c\ \delta f + 4\ b^2\ t\ \delta c\ \delta f - 18\ p_A - 6\ \delta c\ p_A + 10\ \delta f\ p_A + 9\ p_B + 3\ \delta c\ p_B - 10\ \delta f\ p_B\right)$$

Using the derivative, we obtain the reaction function of Firm A in the range, (a+b)/4<z<(2+a+b)/4:

$$\text{Simplify}\Bigg[\text{Solve}\Bigg[-\frac{1}{2\ (a - b)\ t\ (3 + \delta c)^2}\ \left(-9\ a^2\ t + 9\ b^2\ t - 6\ a\ t\ \delta c + 6\ b\ t\ \delta c - 2\ a\ t\ \delta c^2 + a^2\ t\ \delta c^2 + 2\ b\ t\ \delta c^2 - b^2\ t\ \delta c^2 - 8\ a\ t\ \delta f + 8\ a^2\ t\ \delta f + 8\ b\ t\ \delta f - 8\ b^2\ t\ \delta f + 4\ a\ t\ \delta c\ \delta f - 4\ a^2\ t\ \delta c\ \delta f - 4\ b\ t\ \delta c\ \delta f + 4\ b^2\ t\ \delta c\ \delta f - 18\ p_A - 6\ \delta c\ p_A + 10\ \delta f\ p_A + 9\ p_B + 3\ \delta c\ p_B - 10\ \delta f\ p_B\right) == 0,\ p_A\Bigg]\Bigg]$$

$$\left\{\left\{p_A \to \frac{1}{2\ (9 + 3\ \delta c - 5\ \delta f)}\ \left((a - b)\ t\ \left(a\ \left(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\right) + b\ \left(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\right) - 2\ \left(3\ \delta c + \delta c^2 + 4\ \delta f - 2\ \delta c\ \delta f\right)\right) + (9 + 3\ \delta c - 10\ \delta f)\ p_B\right)\right\}\right\}$$

The function might be outside the range, (a+b)/4<z<(2+a+b)/4.
We derive the condition that the reaction function is indeed in the range, (a+b)/4<z<(2+a+b)/4.

We have already obtained the condition that z is between (a+b)/4<z<(2+a+b)/4 as follows:

$$p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) - 2\ (3 - \delta c))}{6} < p_A < p_B + \frac{(b - a)\ t\ (3\ (a + b)\ (1 - \delta c) + 4\ \delta c)}{6}$$

If the following outcomes are positive, the reaction function satisfies the above inequalities:

```
Simplify[pB + (b - a) t (3 (a + b) (1 - δc) + 4 δc)/6 - 1/(2 (9 + 3 δc - 5 δf))
  ((a - b) t (a (-9 + δc^2 + 8 δf - 4 δc δf) + b (-9 + δc^2 + 8 δf - 4 δc δf) - 2 (3 δc + δc^2 + 4 δf - 2 δc δf)) +
   (9 + 3 δc - 10 δf) pB)]
Simplify[1/(2 (9 + 3 δc - 5 δf)) ((a - b) t (a (-9 + δc^2 + 8 δf - 4 δc δf) +
      b (-9 + δc^2 + 8 δf - 4 δc δf) - 2 (3 δc + δc^2 + 4 δf - 2 δc δf)) +
   (9 + 3 δc - 10 δf) pB) - (pB + (b - a) t (3 (a + b) (1 - δc) - 2 (3 - δc))/6)]
```

$$\frac{(3+\delta c)\ ((a-b)\ t\ (6\ (-1+a+b)\ \delta c+(8-3\ a-3\ b)\ \delta f)+9\ p_B)}{6\ (9+3\ \delta c-5\ \delta f)}$$

$$-(((3+\delta c)\ ((a-b)\ t\ (18+6\ a\ \delta c+6\ b\ \delta c-2\ \delta f-3\ a\ \delta f-3\ b\ \delta f)+9\ p_B))\ /\ (6\ (9+3\ \delta c-5\ \delta f)))$$

We derive the threshold values of p_A such that each of the outcomes equals zero:

```
Simplify[Solve[(3 + δc) ((a - b) t (6 (-1 + a + b) δc + (8 - 3 a - 3 b) δf) + 9 pB)/(6 (9 + 3 δc - 5 δf)) == 0, pB]]
Simplify[Solve[
  -(((3 + δc) ((a - b) t (18 + 6 a δc + 6 b δc - 2 δf - 3 a δf - 3 b δf) + 9 pB)) / (6 (9 + 3 δc - 5 δf))) == 0,
  pB]]
```

$$\left\{\left\{p_B \to -\frac{1}{9}\ (a-b)\ t\ (6\ (-1+a+b)\ \delta c+(8-3\ a-3\ b)\ \delta f)\right\}\right\}$$

$$\left\{\left\{p_B \to -\frac{1}{9}\ (a-b)\ t\ (18+6\ a\ \delta c+6\ b\ \delta c-2\ \delta f-3\ a\ \delta f-3\ b\ \delta f)\right\}\right\}$$

Therefore, if p_B satisfies the following inequalities, the reaction function of Firm B is in the range, (a+b)/4<z<(2+a+b)/4.

$$\frac{(b-a)\ t\ (6\ (-1+a+b)\ \delta c+(8-3\ a-3\ b)\ \delta f)}{9} < p_B < \frac{(b-a)\ t\ (3\ (a+b)\ (2\ \delta c-\delta f)+2\ (9-\delta f))}{9}$$

Similarly, using the above outcomes, we derive the first-order derivative of Firm B's profit with respect to p_B

```
Factor[D[pB (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2), pB]]
```

$$\frac{1}{2\ (a-b)\ t\ (3+\delta c)^2}\Big(18\ a\ t-9\ a^2\ t-18\ b\ t+9\ b^2\ t+6\ a\ t\ \delta c-6\ b\ t\ \delta c+a^2\ t\ \delta c^2-b^2\ t\ \delta c^2-8\ a\ t\ \delta f+8\ a^2\ t\ \delta f+8\ b\ t\ \delta f-8\ b^2\ t\ \delta f+4\ a\ t\ \delta c\ \delta f-4\ a^2\ t\ \delta c\ \delta f-4\ b\ t\ \delta c\ \delta f+4\ b^2\ t\ \delta c\ \delta f-9\ p_A-3\ \delta c\ p_A+10\ \delta f\ p_A+18\ p_B+6\ \delta c\ p_B-10\ \delta f\ p_B\Big)$$

Using the derivative, we obtain the reaction function of Firm B in the range, (a+b)/4<z<(2+a+b)/4.

```
Simplify[
 Solve[1/(2 (a - b) t (3 + δc)^2) (18 a t - 9 a^2 t - 18 b t + 9 b^2 t + 6 a t δc - 6 b t δc + a^2 t δc^2 - b^2 t δc^2 -
     8 a t δf + 8 a^2 t δf + 8 b t δf - 8 b^2 t δf + 4 a t δc δf - 4 a^2 t δc δf - 4 b t δc δf +
     4 b^2 t δc δf - 9 pA - 3 δc pA + 10 δf pA + 18 pB + 6 δc pB - 10 δf pB) == 0, pB]]
```

$$\left\{\left\{p_B \to \frac{1}{2\ (9+3\ \delta c-5\ \delta f)}\Big(-(a-b)\ t\ \big(18+6\ \delta c-8\ \delta f+4\ \delta c\ \delta f+a\ (-9+\delta c^2+8\ \delta f-4\ \delta c\ \delta f)+b\ (-9+\delta c^2+8\ \delta f-4\ \delta c\ \delta f)\big)+(9+3\ \delta c-10\ \delta f)\ p_A\Big)\right\}\right\}$$

We have already obtained the condition that *z* is between (a+b)/4 and (2+a+b)/4 as follows:

$$p_B + \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) - 2\ (3-\delta c))}{6} < p_A < p_B + \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) + 4\ \delta c)}{6}$$

which can be rewritten as

$$p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) + 4\ \delta c)}{6} < p_B < p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) - 2\ (3-\delta c))}{6}$$

If the following outcomes are positive, the reaction function satisfies the above inequalities:

$$\text{Factor}\Big[p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) - 2\ (3-\delta c))}{6} - \frac{1}{2\ (9+3\ \delta c - 5\ \delta f)} \Big(-(a-b)\ t\ \big(18 + 6\ \delta c - 8\ \delta f + 4\ \delta c\ \delta f + a\ \big(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\big) + b\ \big(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\big)\big) + (9 + 3\ \delta c - 10\ \delta f)\ p_A\Big)\Big]$$

$$\text{Factor}\Big[\frac{1}{2\ (9+3\ \delta c - 5\ \delta f)} \Big(-(a-b)\ t\ \big(18 + 6\ \delta c - 8\ \delta f + 4\ \delta c\ \delta f + a\ \big(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\big) + b\ \big(-9 + \delta c^2 + 8\ \delta f - 4\ \delta c\ \delta f\big)\big) + (9 + 3\ \delta c - 10\ \delta f)\ p_A\Big) - \Big(p_A - \frac{(b-a)\ t\ (3\ (a+b)\ (1-\delta c) + 4\ \delta c)}{6}\Big)\Big]$$

$$-\Big(\Big((3+\delta c)\ \big(-6\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 6\ b\ t\ \delta c - 6\ b^2\ t\ \delta c - 2\ a\ t\ \delta f - 3\ a^2\ t\ \delta f + 2\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\big)\Big) \Big/ (6\ (9 + 3\ \delta c - 5\ \delta f))\Big)$$

$$\frac{1}{6\ (9+3\ \delta c - 5\ \delta f)}\ (3+\delta c)\ \big(-18\ a\ t + 18\ b\ t - 12\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 12\ b\ t\ \delta c - 6\ b^2\ t\ \delta c + 8\ a\ t\ \delta f - 3\ a^2\ t\ \delta f - 8\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\big)$$

We derive the threshold values of p_A such that each of the outcomes equals zero:

$$\text{Simplify}\Big[\text{Solve}\Big[-\Big(\Big((3+\delta c)\ \big(-6\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 6\ b\ t\ \delta c - 6\ b^2\ t\ \delta c - 2\ a\ t\ \delta f - 3\ a^2\ t\ \delta f + 2\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\big)\Big) \Big/ (6\ (9 + 3\ \delta c - 5\ \delta f))\Big) == 0,\ p_A\Big]\Big]$$

$$\text{Simplify}\Big[\text{Solve}\Big[\frac{1}{6\ (9+3\ \delta c - 5\ \delta f)}\ (3+\delta c)\ \big(-18\ a\ t + 18\ b\ t - 12\ a\ t\ \delta c + 6\ a^2\ t\ \delta c + 12\ b\ t\ \delta c - 6\ b^2\ t\ \delta c + 8\ a\ t\ \delta f - 3\ a^2\ t\ \delta f - 8\ b\ t\ \delta f + 3\ b^2\ t\ \delta f - 9\ p_A\big) == 0,\ p_A\Big]\Big]$$

$$\Big\{\Big\{p_A \to \frac{1}{9}\ (a-b)\ t\ (6\ (-1+a+b)\ \delta c - (2 + 3\ a + 3\ b)\ \delta f)\Big\}\Big\}$$

$$\Big\{\Big\{p_A \to \frac{1}{9}\ (a-b)\ t\ (-18 + 6\ (-2+a+b)\ \delta c + (8 - 3\ a - 3\ b)\ \delta f)\Big\}\Big\}$$

Therefore, if p_A satisfies the following inequalities, the reaction function of Firm B is in the range, (a+b)/4<z<(2+a+b)/4

$$\frac{(b-a)\ t\ (6\ (1-a-b)\ \delta c + (2 + 3\ a + 3\ b)\ \delta f)}{9} < p_A < \frac{(b-a)\ t\ (18 + 6\ (2-a-b)\ \delta c - (8 - 3\ a - 3\ b)\ \delta f)}{9}$$

(Case iii) $(2 + a + b)/4 \le z \le 1$

The following prices are the second period prices.

$$\Big\{p_{Ao} \to -\frac{1}{3}\ (a-b)\ t\ (a+b+2\ z),\ p_{An} \to 0,\ p_{Bo} \to (a-b)\ t\ (a+b-2\ z),\ p_{Bn} \to \frac{1}{3}\ (a-b)\ t\ (a+b-4\ z)\Big\}$$

Anticipating the second period prices, consumers choose one of the first-period prices p_A or p_B (p_A is the first-period price of firm A and p_B is the first-period price of firm B)

The location of the indifferent consumer, *z*, is derived by the following equation:

$$\text{Solve}\left[-p_A - t\,(z-a)^2 - \delta c\left(\frac{1}{3}(a-b)\,t\,(a+b-4z) + t\,(z-b)^2\right) == -p_B - t\,(z-b)^2 - \delta c\left(((a-b)\,t\,(a+b-2z)) + t\,(z-b)^2\right),\ z\right]$$

$$\left\{\left\{z \to \frac{-3a^2t + 3b^2t + 2a^2t\,\delta c - 2b^2t\,\delta c - 3p_A + 3p_B}{2(a-b)\,t\,(-3+\delta c)}\right\}\right\}$$

We set the location of the indifferent consumer z:

$$z = \frac{-3a^2t + 3b^2t + 2a^2t\,\delta c - 2b^2t\,\delta c - 3p_A + 3p_B}{2(a-b)\,t\,(-3+\delta c)}$$

$$\frac{-3a^2t + 3b^2t + 2a^2t\,\delta c - 2b^2t\,\delta c - 3p_A + 3p_B}{2(a-b)\,t\,(-3+\delta c)}$$

We derive the condition that the location of the indifferent consumer, z, is between (2+a+b)/4 and 1, by solving the following equations

```
Factor[Solve[z - (2 + a + b) / 4 == 0, pA]]
Factor[Solve[1 - z == 0, pA]]
```

$$\left\{\left\{p_A \to \frac{1}{6}\left(6at - 3a^2t - 6bt + 3b^2t - 2at\,\delta c + 3a^2t\,\delta c + 2bt\,\delta c - 3b^2t\,\delta c + 6p_B\right)\right\}\right\}$$

$$\left\{\left\{p_A \to \frac{1}{3}\left(6at - 3a^2t - 6bt + 3b^2t - 2at\,\delta c + 2a^2t\,\delta c + 2bt\,\delta c - 2b^2t\,\delta c + 3p_B\right)\right\}\right\}$$

By simplifying the above values of p_A, we have the condition that z is between (2+a+b)/4 and 1 as follows:

$$p_B + \frac{(b-a)\,t\,((a+b)(3-2\delta c) - 2(3-\delta c))}{3} \le p_A \le p_B + \frac{(b-a)\,t\,(3(a+b)(1-\delta c) - 2(3-\delta c))}{6}$$

We have already derived the 2nd period profits of Firms A and B as follows:

$$\pi_{A2} : -\frac{1}{18}(a-b)\,t\,(a+b+2z)^2$$

$$\pi_{B2} : -\frac{1}{18}(a-b)\,t\left(-18a + a^2 - 18b + 2ab + b^2 + 36z + 10az + 10bz - 20z^2\right)$$

The first-order condition of Firm A with respect to p_A is

$$\text{Factor}\left[D\left[p_A z + \delta f\left(\frac{1}{18}(b-a)\,t\,(a+b+2z)^2\right),\ p_A\right]\right]$$

$$\frac{1}{2(a-b)\,t\,(-3+\delta c)^2}\left(9a^2t - 9b^2t - 9a^2t\,\delta c + 9b^2t\,\delta c + 2a^2t\,\delta c^2 - 2b^2t\,\delta c^2 - 4a^2t\,\delta f + 4b^2t\,\delta f + 2a^2t\,\delta c\,\delta f - 2b^2t\,\delta c\,\delta f + 18p_A - 6\delta c\,p_A - 2\delta f\,p_A - 9p_B + 3\delta c\,p_B + 2\delta f\,p_B\right)$$

The reaction function of Firm A within the range in which (2 + a + b)/4≤z≤1 is

$$\text{Simplify}\left[\text{Solve}\left[\frac{1}{2(a-b)\,t\,(-3+\delta c)^2}\left(9a^2t - 9b^2t - 9a^2t\,\delta c + 9b^2t\,\delta c + 2a^2t\,\delta c^2 - 2b^2t\,\delta c^2 - 4a^2t\,\delta f + 4b^2t\,\delta f + 2a^2t\,\delta c\,\delta f - 2b^2t\,\delta c\,\delta f + 18p_A - 6\delta c\,p_A - 2\delta f\,p_A - 9p_B + 3\delta c\,p_B + 2\delta f\,p_B\right) == 0,\ p_A\right]\right]$$

$$\left\{\left\{p_A \to \left(\left(a^2-b^2\right)t\left(9 + 2\delta c^2 - 4\delta f + \delta c\,(-9+2\delta f)\right) + (-9 + 3\delta c + 2\delta f)\,p_B\right) / \left(2(-9 + 3\delta c + \delta f)\right)\right\}\right\}$$

$$p_A \to \frac{(9 - 3\delta c - 2\delta f)\,p_B}{2(9 - 3\delta c - \delta f)} + \frac{(b-a)(a+b)\,t\,((3-\delta c)(3-2\delta c) - 2(2-\delta c)\,\delta f)}{2(9 - 3\delta c - \delta f)}$$

The function might be outside the range, (2 + a + b)/4≤z≤1.
We derive the condition that the reaction function is indeed in the range, (2 + a + b)/4≤z≤1.

We have already obtained the condition that z is between (2+a + b)/4 and 1 as follows:

$$p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3} \le p_A \le p_B + \frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6}$$

If p_B is larger than $p_A - \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}$, z becomes 1.

In this case, Firm A chooses the following p_A which just leads to z=1.

$$p_A \to p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

The reaction function is within the range (2 + a + b)/4≤z≤1 if the following are positive:

$$\text{Simplify}\Big[\text{Factor}\Big[\Big(\frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)} + \frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\Big) - \Big(p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}\Big)\Big]\Big]$$

$$\text{Simplify}\Big[\text{Factor}\Big[p_B + \frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6} - \Big(\frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)} + \frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\Big)\Big]\Big]$$

$$-(((-3+\delta c)\,((a-b)\,t\,(a\,(-9+6\,\delta c-2\,\delta f)+b\,(-9+6\,\delta c-2\,\delta f)-4\,(-9+3\,\delta c+\delta f))+9\,p_B))\,/\,(6\,(-9+3\,\delta c+\delta f)))$$

$$((-3+\delta c)\,((a-b)\,t\,(18+3\,(-2+a+b)\,\delta c-(2+3\,a+3\,b)\,\delta f)+9\,p_B))\,/\,(6\,(-9+3\,\delta c+\delta f))$$

Therefore, the reaction function of Firm A is within the range (2 + a + b)/4≤z≤1 if

$$\frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9} \ge$$

$$p_B \ge \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}$$

Note that if p_B is larger than the left-hand side value of the inequality, Firm A chooses the following p_A, which leads to z=1.

$$p_A \to p_B + \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

Similarly, using the above outcomes, we derive the first-order condition of Firm B with respect to p_B

$$\text{Factor}\Big[D\Big[p_B\,(1-z) + \delta f\,\Big(\frac{1}{18}\,(b-a)\,t\,\big(-18\,a+a^2-18\,b+2\,a\,b+b^2+36\,z+10\,a\,z+10\,b\,z-20\,z^2\big)\Big),\ p_B\Big]\Big]$$

$$-\frac{1}{2\,(a-b)\,t\,(-3+\delta c)^2}\big(-18\,a\,t+9\,a^2\,t+18\,b\,t-9\,b^2\,t+12\,a\,t\,\delta c-9\,a^2\,t\,\delta c-12\,b\,t\,\delta c+9\,b^2\,t\,\delta c-2\,a\,t\,\delta c^2+2\,a^2\,t\,\delta c^2+2\,b\,t\,\delta c^2-2\,b^2\,t\,\delta c^2-18\,a\,t\,\delta f+5\,a^2\,t\,\delta f+18\,b\,t\,\delta f-5\,b^2\,t\,\delta f+6\,a\,t\,\delta c\,\delta f-5\,a^2\,t\,\delta c\,\delta f-6\,b\,t\,\delta c\,\delta f+5\,b^2\,t\,\delta c\,\delta f+9\,p_A-3\,\delta c\,p_A+10\,\delta f\,p_A-18\,p_B+6\,\delta c\,p_B-10\,\delta f\,p_B\big)$$

The reaction function of Firm B is

$$\text{Simplify}\Big[\text{Solve}\Big[-\frac{1}{2\,(a-b)\,t\,(-3+\delta c)^2}\big(-18\,a\,t+9\,a^2\,t+18\,b\,t-9\,b^2\,t+12\,a\,t\,\delta c-9\,a^2\,t\,\delta c-12\,b\,t\,\delta c+9\,b^2\,t\,\delta c-2\,a\,t\,\delta c^2+2\,a^2\,t\,\delta c^2+2\,b\,t\,\delta c^2-2\,b^2\,t\,\delta c^2-18\,a\,t\,\delta f+5\,a^2\,t\,\delta f+18\,b\,t\,\delta f-5\,b^2\,t\,\delta f+6\,a\,t\,\delta c\,\delta f-5\,a^2\,t\,\delta c\,\delta f-6\,b\,t\,\delta c\,\delta f+5\,b^2\,t\,\delta c\,\delta f+9\,p_A-3\,\delta c\,p_A+10\,\delta f\,p_A-18\,p_B+6\,\delta c\,p_B-10\,\delta f\,p_B\big)==0,\ p_B\Big]\Big]$$

$$\Big\{\Big\{p_B\to\frac{1}{2\,(-9+3\,\delta c-5\,\delta f)}\Big(-(a-b)\,t\,\big(-2\,(-3+\delta c)\,(-3+\delta c-3\,\delta f)+b\,(9-9\,\delta c+2\,\delta c^2+5\,\delta f-5\,\delta c\,\delta f)+a\,(9+2\,\delta c^2+5\,\delta f-\delta c\,(9+5\,\delta f))\big)+(-9+3\,\delta c-10\,\delta f)\,p_A\Big)\Big\}\Big\}$$

$$p_B\to\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))$$

We check the condition that the reaction function of Firm B is within the range (2 + a + b)/4≤z≤ 1.

$$p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}<p_A<p_B+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6}$$

Note that if p_B is larger than $p_A-\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}$, z becomes 1.

The reaction function of Firm B is within the range in which (2 + a + b)/4≤z≤1 if the following are positive:

$$\text{Simplify}\Big[\text{Factor}\Big[p_A-\Big(\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}\Big)\Big]\Big]$$

$$\text{Simplify}\Big[\text{Factor}\Big[\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))+\frac{(b-a)\,t\,(3\,(a+b)\,(1-\delta c)-2\,(3-\delta c))}{6}-p_A\Big]\Big]$$

$$-(((-3+\delta c)\,((a-b)\,t\,(a\,(-9+6\,\delta c-5\,\delta f)+b\,(-9+6\,\delta c-5\,\delta f)+2\,(9-3\,\delta c+\delta f))-9\,p_A))\,/\,(6\,(-9+3\,\delta c-5\,\delta f)))$$

$$\frac{(-3+\delta c)\,((a-b)\,t\,(3\,a\,\delta c+3\,b\,\delta c-8\,\delta f)-9\,p_A)}{6\,(-9+3\,\delta c-5\,\delta f)}$$

$$\text{Simplify}[\text{Solve}[-(((-3+\delta c)\,((a-b)\,t\,(a\,(-9+6\,\delta c-5\,\delta f)+b\,(-9+6\,\delta c-5\,\delta f)+2\,(9-3\,\delta c+\delta f))-9\,p_A))\,/\,(6\,(-9+3\,\delta c-5\,\delta f)))==0,\ p_A]]$$

$$\text{Simplify}\Big[\text{Solve}\Big[\frac{(-3+\delta c)\,((a-b)\,t\,(3\,a\,\delta c+3\,b\,\delta c-8\,\delta f)-9\,p_A)}{6\,(-9+3\,\delta c-5\,\delta f)}==0,\ p_A\Big]\Big]$$

$$\Big\{\Big\{p_A\to\frac{1}{9}\,(a-b)\,t\,(a\,(-9+6\,\delta c-5\,\delta f)+b\,(-9+6\,\delta c-5\,\delta f)+2\,(9-3\,\delta c+\delta f))\Big\}\Big\}$$

$$\Big\{\Big\{p_A\to\frac{1}{9}\,(a-b)\,t\,(3\,a\,\delta c+3\,b\,\delta c-8\,\delta f)\Big\}\Big\}$$

Thus, the reaction function of Firm B is within the range (2 + a + b)/4≤z≤1 if

$$\frac{(b-a)\,t\,((a+b)\,(9-6\,\delta c+5\,\delta f)-2\,(9-3\,\delta c+\delta f))}{9}\le p_A\le\frac{(b-a)\,t\,(8\,\delta f-3\,(a+b)\,\delta c)}{9}$$

We summarize below the results from our analysis of the reaction functions in the three cases.

(Case i) 0≤z ≤ (a + b)/4.

Firm A's reaction function in this range is

$$p_A \to \left((b-a)\ t\left(6\,\delta c - 2\,\delta c^2 + 8\,\delta f + 4\,\delta c\,\delta f + (a+b)\left(9 - 9\,\delta c + 2\,\delta c^2 + 5\,\delta f - 5\,\delta c\,\delta f\right)\right) + (9 - 3\,\delta c + 10\,\delta f)\ p_B\right) / \left(2\ (9 - 3\,\delta c + 5\,\delta f)\right)$$

This is applicable for the following range of p_B

$$-\frac{1}{9}(b-a)\ t\ ((a+b)\ (9 - 6\,\delta c + 5\,\delta f) + 6\,\delta c - 8\,\delta f) < p_B < \frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c + 8\,\delta f)}{9}$$

We check the endpoints of Firm A's reaction function in the range of p_B (see the inequalities above). Substituting the minimum and maximum values of p_B into Firm A's reaction function, we obtain the endpoints (vectors) of Firm A's reaction function in the $(p_A,\ p_B)$ coordinate system:

```
FullSimplify[
 ((b - a) t (6 δc - 2 δc^2 + 8 δf + 4 δc δf + (a + b) (9 - 9 δc + 2 δc^2 + 5 δf - 5 δc δf)) + (9 - 3 δc + 10 δf)
     pB) / (2 (9 - 3 δc + 5 δf)) /. pB → -1/9 (b - a) t ((a + b) (9 - 6 δc + 5 δf) + 6 δc - 8 δf) ]
FullSimplify[((b - a) t (6 δc - 2 δc^2 + 8 δf + 4 δc δf + (a + b) (9 - 9 δc + 2 δc^2 + 5 δf - 5 δc δf)) +
     (9 - 3 δc + 10 δf) pB) / (2 (9 - 3 δc + 5 δf)) /. pB → ((b - a) t (3 (-2 + a + b) δc + 8 δf))/9]
```

$$\frac{1}{9}(a-b)\ (-8+5\,a+5\,b)\ t\ \delta f$$

$$\frac{1}{18}(a-b)\ t\ (3\ (a+b)\ (-3+\delta c) - 16\,\delta f)$$

Firm A's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{1}{9}(a-b)\ (-8+5\,a+5\,b)\ t\,\delta f,\ -\frac{1}{9}(b-a)\ t\ ((a+b)\ (9-6\,\delta c+5\,\delta f)+6\,\delta c-8\,\delta f)\right),$$

$$\left(\frac{1}{18}(a-b)\ t\ (3\ (a+b)\ (-3+\delta c)-16\,\delta f),\ \frac{(b-a)\ t\ (3\ (-2+a+b)\ \delta c+8\,\delta f)}{9}\right)$$

Firm B's reaction function in this range is

$$p_B \to \frac{1}{2\ (9 - 3\,\delta c - \delta f)}\left(((b-a)\ (2-a-b))\ t\left(9 + 2\,\delta c^2 - 4\,\delta f + \delta c\ (-9 + 2\,\delta f)\right) + (9 - 3\,\delta c - 2\,\delta f)\ p_A\right)$$

This is applicable for the following range of p_A

$$\frac{(b-a)\ t\ (18 + (a+b)\ (9 - 6\,\delta c + 2\,\delta f) - 8\,\delta f)}{9} \geq p_A \geq \frac{(b-a)\ t\ (18 - 3\ (a+b)\ (\delta c - \delta f) - 8\,\delta f)}{9}$$

Note that if p_A is larger than the left-hand side value of the inequality, Firm B chooses the following p_B, which just leads to z=0.

$$p_B \to p_A - \frac{(b-a)\ t\ ((a+b)\ (3 - 2\,\delta c) + 2\,\delta c)}{3}$$

We check the endpoints of Firm B's reaction function in the range of p_A, based on the above inequalities. Substituting the minimum and maximum values of p_A into Firm B's reaction function, we obtain the endpoints (vectors) of Firm B's reaction function in the (p_A, p_B) coordinate system:

$$\text{FullSimplify}\Big[\frac{1}{2\,(9-3\,\delta c-\delta f)}\Big(((b-a)\,(2-a-b))\,t\,\big(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\big)+(9-3\,\delta c-2\,\delta f)\,p_A\Big)\ /.\ p_A\to\frac{(b-a)\,t\,(18-3\,(a+b)\,(\delta c-\delta f)-8\,\delta f)}{9}\Big]$$

$$\text{FullSimplify}\Big[\frac{1}{2\,(9-3\,\delta c-\delta f)}\Big(((b-a)\,(2-a-b))\,t\,\big(9+2\,\delta c^2-4\,\delta f+\delta c\,(-9+2\,\delta f)\big)+(9-3\,\delta c-2\,\delta f)\,p_A\Big)\ /.\ p_A\to\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}\Big]$$

$$-\frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(-3+\delta c)+2\,(-8+3\,a+3\,b)\,\delta f)$$

$$-\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)$$

Firm B's reaction function in this range consists of the line segment between the following two points:

$$\Big(\frac{(b-a)\,t\,(18-3\,(a+b)\,(\delta c-\delta f)-8\,\delta f)}{9},\ -\frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(-3+\delta c)+2\,(-8+3\,a+3\,b)\,\delta f)\Big),$$

$$\Big(\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9},\ -\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)\Big)$$

Note that if the following inequality,

$$\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}\le p_A,$$

holds, then Firm B chooses the following p_B, which just leads to z=0.

$$p_B\to p_A-\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)+2\,\delta c)}{3}.$$

The reaction function of firm B leading to z=0 consists of the line segment connecting the following two points.

$$\Big(\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9},\ -\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)\Big),$$

$$\Big(\frac{(b-a)\,t\,(18+(a+b)\,(9-6\,\delta c+2\,\delta f)-8\,\delta f)}{9}+k,\ -\frac{2}{9}\,(a-b)\,t\,(9-3\,\delta c+(-4+a+b)\,\delta f)+k\Big),$$

where *k* is a (sufficiently large) positive constant.

(Case ii) (a + b)/4 < z < (2 + a + b)/4

Firm A's reaction function in this range is

$$p_A\to\Big((9+3\,\delta c-10\,\delta f)\,p_B+(b-a)\,t\,\big((a+b)\,\big(9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f\big)+2\,\big(3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f\big)\big)\Big)\Big/(2\,(9+3\,\delta c-5\,\delta f))$$

This is applicable for the following range of p_B

$$\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)-2\,(3\,\delta c-4\,\delta f))}{9}<p_B<\frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}$$

We check the endpoints of Firm A's reaction function in the range of p_B (see the inequalities above). Substituting the minimum and maximum values of p_B into Firm A's reaction function, we obtain the endpoints (vectors) of Firm A's reaction function in the (p_A, p_B) coordinate system:

```
FullSimplify[
 ((9 + 3 δc - 10 δf) pB + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
   (2 (9 + 3 δc - 5 δf)) /. pB → ((b - a) t (3 (a + b) (2 δc - δf) - 2 (3 δc - 4 δf)))/9]
FullSimplify[((9 + 3 δc - 10 δf) pB +
     (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
   (2 (9 + 3 δc - 5 δf)) /. pB → ((b - a) t (3 (a + b) (2 δc - δf) + 2 (9 - δf)))/9]
```

$$-\frac{1}{18} (a-b)\, t\, (3 (a+b) (3+\delta c) - 2 (-8+3 a+3 b)\, \delta f)$$

$$-\frac{1}{18} (a-b)\, t\, (3 (2+a+b) (3+\delta c) - 2 (2+3 a+3 b)\, \delta f)$$

Firm A's reaction function in this range consists of the line segment between the following two points:

$$\left(-\frac{1}{18} (a-b)\, t\, (3 (a+b) (3+\delta c) - 2 (-8+3 a+3 b)\, \delta f),\ \frac{(b-a)\, t\, (3 (a+b) (2 \delta c-\delta f) - 2 (3 \delta c-4 \delta f))}{9}\right),$$

$$\left(-\frac{1}{18} (a-b)\, t\, (3 (2+a+b) (3+\delta c) - 2 (2+3 a+3 b)\, \delta f),\ \frac{(b-a)\, t\, (3 (a+b) (2 \delta c-\delta f) + 2 (9-\delta f))}{9}\right)$$

Firm B's reaction function in this range is

$$p_B \to \left((b-a)\, t \left(2 (9+3 \delta c-4 \delta f+2 \delta c\, \delta f) + (a+b) \left(-9+\delta c^2+8 \delta f-4 \delta c\, \delta f\right)\right) + (9+3 \delta c-10 \delta f)\, p_A\right) \Big/ (2 (9+3 \delta c-5 \delta f))$$

This is applicable for the following range of p_A

$$\frac{(b-a)\, t\, (6 (1-a-b)\, \delta c + (2+3 a+3 b)\, \delta f)}{9} < p_A < \frac{(b-a)\, t\, (18+6 (2-a-b)\, \delta c - (8-3 a-3 b)\, \delta f)}{9}$$

We check the endpoints of Firm B's reaction function in the range of p_A (see the inequalities above). Substituting the minimum and maximum values of p_A into Firm B's reaction function, we obtain the endpoints (vectors) of Firm B's reaction function in the (p_A, p_B) coordinate system:

```
FullSimplify[
 ((b - a) t (2 (9 + 3 δc - 4 δf + 2 δc δf) + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf)) + (9 + 3 δc - 10 δf) pA) /
   (2 (9 + 3 δc - 5 δf)) /. pA → ((b - a) t (6 (1 - a - b) δc + (2 + 3 a + 3 b) δf))/9]
FullSimplify[((b - a) t (2 (9 + 3 δc - 4 δf + 2 δc δf) + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf)) +
     (9 + 3 δc - 10 δf) pA) / (2 (9 + 3 δc - 5 δf)) /.
  pA → ((b - a) t (18 + 6 (2 - a - b) δc - (8 - 3 a - 3 b) δf))/9]
```

$$\frac{1}{18} (a-b)\, t\, (3 (-2+a+b) (3+\delta c) - 2 (2+3 a+3 b)\, \delta f)$$

$$\frac{1}{18} (a-b)\, t\, (3 (-4+a+b) (3+\delta c) - 2 (-8+3 a+3 b)\, \delta f)$$

Firm B's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{(b-a)\,t\,(6\,(1-a-b)\,\delta c+(2+3a+3b)\,\delta f)}{9},\ \frac{1}{18}\,(a-b)\,t\,(3\,(-2+a+b)\,(3+\delta c)-2\,(2+3a+3b)\,\delta f)\right),$$

$$\left(\frac{(b-a)\,t\,(18+6\,(2-a-b)\,\delta c-(8-3a-3b)\,\delta f)}{9},\ \frac{1}{18}\,(a-b)\,t\,(3\,(-4+a+b)\,(3+\delta c)-2\,(-8+3a+3b)\,\delta f)\right)$$

(Case iii) $(2 + a + b)/4 \leq 1 \leq z$

Firm A's reaction function in this range is

$$p_A \to \frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)}+\frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}$$

This is applicable for the following range of p_B

$$\frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9} \geq$$

$$p_B \geq \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}$$

Note that if p_B is larger than the left-hand side value of the inequality, Firm A chooses the following p_A, which leads to z=1.

$$p_A \to p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

We check the endpoints of Firm A's reaction function in the range of p_B (see the inequalities above). Substituting the minimum and maximum values of p_B into Firm A's reaction function, we obtain the endpoints (vectors) of Firm A's reaction function in the $(p_A,\ p_B)$ coordinate system:

$$\text{FullSimplify}\left[\frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)}+\frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\ /.\ p_B \to \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}\right]$$

$$\text{FullSimplify}\left[\frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)}+\frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\ /.\ p_B \to \frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}\right]$$

$$\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta c)+2\,(2+3a+3b)\,\delta f)$$

$$\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f)$$

Firm A's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta c)+2\,(2+3a+3b)\,\delta f),\ \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}\right),$$

$$\left(\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f),\ \frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}\right)$$

Note that if the following inequality,

$$\frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}\le p_B,$$

holds, then Firm A chooses the following p_A, which leads to z=1.

$$p_A \to p_B+\frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)-2\,(3-\delta c))}{3}.$$

The reaction function of firm A leading to z=1 consists of the line segment connecting the following two points.

$$\left(\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f),\ \frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}\right),$$
$$\left(\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f)+k,\ \frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}+k\right),$$

where k is a (sufficiently large) positive constant.

Firm B's reaction function in this range is

$$p_B\to\frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+ ((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))$$

This is applicable for the following range of p_A

$$\frac{(b-a)\,t\,((a+b)\,(9-6\,\delta c+5\,\delta f)-2\,(9-3\,\delta c+\delta f))}{9}<p_A<\frac{(b-a)\,t\,(8\,\delta f-3\,(a+b)\,\delta c)}{9}$$

We check the endpoints of Firm B's reaction function in the range of p_A (see the inequalities above). Substituting the minimum and maximum values of p_A into Firm B's reaction function, we obtain the endpoints (vectors) of Firm B's reaction function in the (p_A, p_B) coordinate system:

```
FullSimplify[(9 - 3 δc + 10 δf) pA / (2 (9 - 3 δc + 5 δf)) +
  ((b - a) t (2 (3 - δc) (3 - δc + 3 δf) - (a + b) ((3 - δc) (3 - 2 δc) + 5 (1 - δc) δf))) /
   (2 (9 - 3 δc + 5 δf)) /. pA → (b - a) t ((a + b) (9 - 6 δc + 5 δf) - 2 (9 - 3 δc + δf)) / 9]
FullSimplify[(9 - 3 δc + 10 δf) pA / (2 (9 - 3 δc + 5 δf)) +
  ((b - a) t (2 (3 - δc) (3 - δc + 3 δf) - (a + b) ((3 - δc) (3 - 2 δc) + 5 (1 - δc) δf))) /
   (2 (9 - 3 δc + 5 δf)) /. pA → (b - a) t (8 δf - 3 (a + b) δc) / 9]
```

$$-\frac{1}{9}\,(a-b)\,(-2+5\,a+5\,b)\,t\,\delta f$$

$$-\frac{1}{18}\,(a-b)\,t\,(3\,(-2+a+b)\,(-3+\delta c)+16\,\delta f)$$

Firm B's reaction function in this range consists of the line segment between the following two points:

$$\left(\frac{(b-a)\,t\,((a+b)\,(9-6\,\delta c+5\,\delta f)-2\,(9-3\,\delta c+\delta f))}{9},\ -\frac{1}{9}\,(a-b)\,(-2+5\,a+5\,b)\,t\,\delta f\right),$$
$$\left(\frac{(b-a)\,t\,(8\,\delta f-3\,(a+b)\,\delta c)}{9},\ -\frac{1}{18}\,(a-b)\,t\,(3\,(-2+a+b)\,(-3+\delta c)+16\,\delta f)\right)$$

Because each firm's reaction function consists of three different pieces, we need to derive the 'true' reaction function by checking when the firm's profit obtains a global, rather than, local maximum.

From here on, we assume that the discount factors are common δ. We set the following :

δf = δ

δ

δc = δ

δ

From Firm A's reaction function derived above, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in period 1.

$$\text{c1aL} = \text{Simplify}\left[\left\{\frac{1}{9}\,(a-b)\,(-8+5a+5b)\,t\,\delta f,\ -\frac{1}{9}\,(b-a)\,t\,((a+b)\,(9-6\,\delta c+5\,\delta f)+6\,\delta c-8\,\delta f)\right\}\right]$$

$$\text{c1aR} = \text{FullSimplify}\left[\text{Factor}\left[\left\{\frac{1}{18}\,(a-b)\,t\,(3\,(a+b)\,(-3+\delta c)-16\,\delta f),\ \frac{(b-a)\,t\,(3\,(-2+a+b)\,\delta c+8\,\delta f)}{9}\right\}\right]\right]$$

$$\left\{\frac{1}{9}\,(a-b)\,(-8+5a+5b)\,t\,\delta,\ -\frac{1}{9}\,(-a+b)\,t\,(-(a+b)\,(-9+\delta)-2\,\delta)\right\}$$

$$\left\{\frac{1}{18}\,(a-b)\,t\,(-9\,(a+b)+(-16+3a+3b)\,\delta),\ -\frac{1}{9}\,(a-b)\,(2+3a+3b)\,t\,\delta\right\}$$

Case (ii): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in period 1.

$$\text{c2aL} = \text{FullSimplify}\left[\text{Factor}\left[\left\{-\frac{1}{18}\,(a-b)\,t\,(3\,(a+b)\,(3+\delta c)-2\,(-8+3a+3b)\,\delta f),\ \frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)-2\,(3\,\delta c-4\,\delta f))}{9}\right\}\right]\right]$$

$$\text{c2aR} = \text{Simplify}\left[\left\{-\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(3+\delta c)-2\,(2+3a+3b)\,\delta f),\ \frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}\right\}\right]$$

$$\left\{\frac{1}{18}\,(a-b)\,t\,(-9\,(a+b)+(-16+3a+3b)\,\delta),\ -\frac{1}{9}\,(a-b)\,(2+3a+3b)\,t\,\delta\right\}$$

$$\left\{\frac{1}{18}\,(a-b)\,t\,(3a\,(-3+\delta)+3b\,(-3+\delta)-2\,(9+\delta)),\ \frac{1}{9}\,(-a+b)\,t\,(18+(-2+3a+3b)\,\delta)\right\}$$

As shown above, the left endpoint of Firm A's reaction function in (Case ii) coincides with the right endpoint of Firm A's reaction function in (Case i). That is, Firm A's reaction function is continuous in (Case i) and (Case ii).

(Case iii): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in period 1.

```
c3aL = Simplify[{1/18 (a - b) t (3 (2 + a + b) (-3 + δc) + 2 (2 + 3 a + 3 b) δf),
   ((b - a) t (2 (9 - 3 δc - δf) + 3 (a + b) (δc - δf)))/9}]
c3aR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf),
   ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
```

$$\left\{\frac{1}{18}(a-b)\,t\,(3(2+a+b)(-3+\delta)+2(2+3a+3b)\delta),\ \frac{2}{9}(a-b)\,t\,(-9+4\delta)\right\}$$

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\delta),\ -\frac{1}{9}(a-b)(-4+a+b)\,t\,(-9+4\delta)\right\}$$

At this stage, we cannot say if Firm A's reaction function is also continuous in (Case ii) and (Case iii). We will come back to this shortly. Note that the left endpoint in (Case iii) corresponds to (2+a+b)/4=z.

(Case iii)': When z=1, the reaction function of Firm A consists of the segment connecting the following two points.

```
c3daL = Simplify[
  {2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf), ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
c3daR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf) + k,
   ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9 + k}]
```

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\delta),\ -\frac{1}{9}(a-b)(-4+a+b)\,t\,(-9+4\delta)\right\}$$

$$\left\{k+\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\delta),\ k-\frac{1}{9}(a-b)(-4+a+b)\,t\,(-9+4\delta)\right\}$$

where k is a sufficient large positive number (to keep p_A at the monopoly price leading to $z=1$).

Next, from Firm B's reaction function derived above, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in period 1.

```
c1bL = Simplify[{(b - a) t (18 - 3 (a + b) (δc - δf) - 8 δf)/9,
  -1/18 (a - b) t (3 (-4 + a + b) (-3 + δc) + 2 (-8 + 3 a + 3 b) δf)}]
c1bR = Simplify[{(b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf)/9,
  -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf)}]
```

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+4\delta),\ -\frac{1}{18}(a-b)\,t\,(36+9a(-1+\delta)+9b(-1+\delta)-28\delta)\right\}$$

$$\left\{\frac{1}{9}(a-b)(2+a+b)\,t\,(-9+4\delta),\ -\frac{2}{9}(a-b)\,t\,(9+(-7+a+b)\delta)\right\}$$

(Case i)’: When z=0, The reaction function of Firm B consists of the segment connecting the following two points.

```
c1dbL =
 Simplify[{(b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf)/9, -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf)}]
c1dbR = Simplify[{(b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf)/9 + k,
  -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf) + k}]
```

$$\left\{\frac{1}{9}(a-b)(2+a+b)\,t\,(-9+4\delta),\ -\frac{2}{9}(a-b)\,t\,(9+(-7+a+b)\delta)\right\}$$

$$\left\{k+\frac{1}{9}(a-b)(2+a+b)\,t\,(-9+4\delta),\ k-\frac{2}{9}(a-b)\,t\,(9+(-7+a+b)\delta)\right\}$$

where *k* is a sufficient large positive number (to keep p_B at the monopoly price leading to $z=0$).

Case (ii): The first endpoint below locates the left-hand side of Firm B’s reaction function, and the second endpoint below locates the right-hand side of Firm B’s reaction function in period 1.

```
c2bL = FullSimplify[{(b - a) t (6 (1 - a - b) δc + (2 + 3 a + 3 b) δf)/9,
  1/18 (a - b) t (3 (-2 + a + b) (3 + δc) - 2 (2 + 3 a + 3 b) δf)}]
c2bR = Simplify[{(b - a) t (18 + 6 (2 - a - b) δc - (8 - 3 a - 3 b) δf)/9,
  1/18 (a - b) t (3 (-4 + a + b) (3 + δc) - 2 (-8 + 3 a + 3 b) δf)}]
```

$$\left\{\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta,\ -\frac{1}{18}(a-b)\,t\,(-9(-2+a+b)+(10+3a+3b)\delta)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\,(-18+(-4+3a+3b)\delta),\ -\frac{1}{18}(a-b)\,t\,(-4(-9+\delta)+3a(-3+\delta)+3b(-3+\delta))\right\}$$

At this stage, we cannot say if Firm B’s reaction function is continuous in (Case i) and (Case ii). We will come back to this shortly. Note that the left-hand endpoint in (Case i) corresponds to (a+b)/4=z.

(Case iii): The first endpoint below locates the left-hand side of Firm B’s reaction function, and the second endpoint below locates the right-hand side of Firm B’s reaction function in period 1.

```
c3bL =
  Simplify[{((b - a) t ((a + b) (9 - 6 δc + 5 δf) - 2 (9 - 3 δc + δf)))/9, -1/9 (a - b) (-2 + 5 a + 5 b) t δf}]
c3bR = FullSimplify[
   Factor[{((b - a) t (8 δf - 3 (a + b) δc))/9, -1/18 (a - b) t (3 (-2 + a + b) (-3 + δc) + 16 δf)}]]
```

$\left\{\frac{1}{9}(a-b)\,t\,(18+a(-9+\delta)+b(-9+\delta)-4\delta),\ -\frac{1}{9}(a-b)(-2+5a+5b)\,t\,\delta\right\}$

$\left\{\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta,\ -\frac{1}{18}(a-b)\,t\,(-9(-2+a+b)+(10+3a+3b)\delta)\right\}$

We find that the left endpoint of Firm B’s reaction function in (Case ii) coincides with the right endpoint of Firm B’s reaction function in (Case iii). That is, Firm B’s reaction function is continuous in (Case ii) and (Case iii).

The exact shape of the above reaction functions depends on various parameters of the model and the values of (a, b) that are given at this stage. In what follows, we show first that, for any values of (a, b), each firm’s reaction function has one discontinuity point. Second, we show that the two reaction functions intersect only in (Case ii) given the restrictions on a + b and δ stated in Proposition 1. This shows that the equilibrium is possible only in (Case ii). We then show that the unique location equilibrium leading to the pricing equilibrium corresponding to (Case ii) is given by a = 0, b = 1.

We start by deriving Firm A’s ‘true’ reaction function illustrating the discontinuity using various examples. We repeat the same for Firm B’s ‘true’ reaction function. Calculations in the two parts are quite messy. If necessary, readers can refer to various figures provided, and skip the calculations to jump directly to the third part, where we show the pricing equilibrium in the first period is possible only in (Case ii).

For illustrative purposes, we start with an example where we set a = 0, b = 1, t = 1, and δ = 1/2.

```
a = 0
b = 1
δ = 1/2
t = 1
k = 1
```

0

1

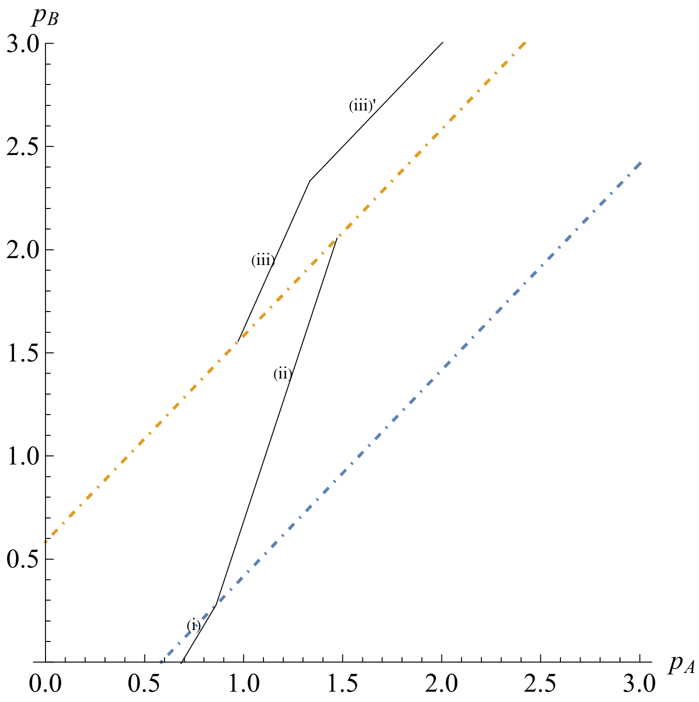
$\frac{1}{2}$

1

1

First, we plot Firm A's reaction function corresponding to the three cases.

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1aR}], Line[{c2aL, c2aR}], , Line[{c3aL, c3aR}],
    Line[{c3daL, c3daR}], Text["(iii)'", {1.6, 2.7}], Text["(iii)", {1.1, 1.95}],
    Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



As shown above, for some p_B, there are two local optimal prices for Firm A.
We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the three endpoints: The right-hand endpoint in (ii) (c2aR), the left-hand and right-hand endpoints (c3aL and c3aR) in (iii) (see below)

```
c2aR
c3aL
c3aR
```

$$\left\{\frac{1}{18}(a-b)\,t\,(3a(-3+\delta)+3b(-3+\delta)-2(9+\delta)),\ \frac{1}{9}(-a+b)\,t\,(18+(-2+3a+3b)\,\delta)\right\}$$

$$\left\{\frac{1}{18}(a-b)\,t\,(3(2+a+b)(-3+\delta)+2(2+3a+3b)\,\delta),\ \frac{2}{9}(a-b)\,t\,(-9+4\delta)\right\}$$

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\,\delta),\ -\frac{1}{9}(a-b)(-4+a+b)\,t\,(-9+4\delta)\right\}$$

First, we compare the elements of the right-hand endpoint in (ii) and the left-hand endpoint in (iii):

```
Factor[1/18 (a - b) t (3 a (-3 + δ) + 3 b (-3 + δ) - 2 (9 + δ)) -
  1/18 (a - b) t (3 (2 + a + b) (-3 + δ) + 2 (2 + 3 a + 3 b) δ)]
Factor[1/9 (-a + b) t (18 + (-2 + 3 a + 3 b) δ) - 2/9 (a - b) t (-9 + 4 δ)]
```

$-\frac{1}{3}(a-b)(2+a+b)\,t\,\delta$

$-\frac{1}{3}(a-b)(2+a+b)\,t\,\delta$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the right-hand endpoint in (ii) is located above the left-hand endpoint in (iii) as in the above Figure.

Second, we compare the p_B-elements of the right-hand endpoint in (ii) and the right-hand endpoint in (iii):

```
Factor[1/9 (-a + b) t (18 + (-2 + 3 a + 3 b) δ) - (-1/9 (a - b) (-4 + a + b) t (-9 + 4 δ))]
```

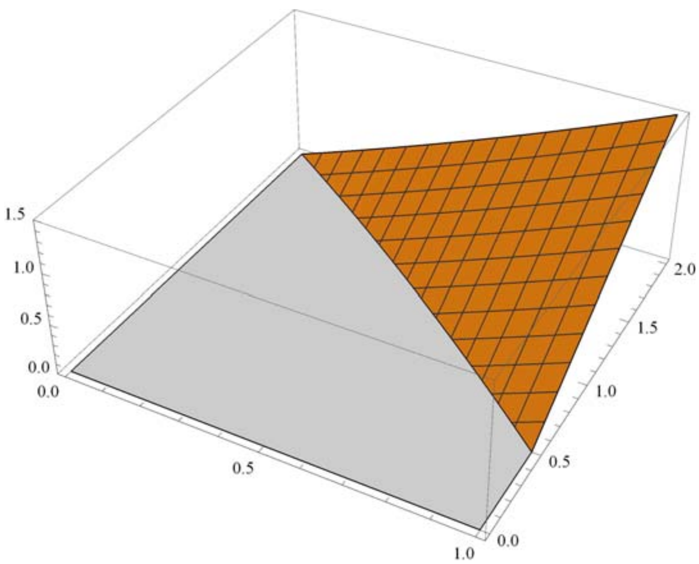
$\frac{1}{9}(a-b)\,t\,(18-9a-9b-14\delta+a\delta+b\delta)$

The p_B-element of the right-hand endpoint in (ii) larger than that of the right-hand endpoint in (iii) if and only if

$\frac{2(9-7\delta)}{9-\delta} < a + b$

If $\frac{2(9-7\delta)}{9-\delta} \geq a+b$, we simply compare the reaction function in (ii) and the reaction function in (iii); If $\frac{2(9-7\delta)}{9-\delta} < a+b$, in addition to the previous comparison, we also compare the reaction function in (ii) and the reaction function in (iii)'.

```
Plot3D[g - (2 (9 - 7 δ))/(9 - δ), {δ, 0, 1}, {g, 0, 2}, PlotRange → {0, 1.5}]
```



We need to find the global optimal price of Firm A, p_A, when there are two local optima for a given p_B. There is a price p_B such that choosing the reaction function in (ii) and choosing the reaction function in (iii) or the reaction function in (iii)' are indifferent for Firm A. This p_B is the threshold for which choosing the reaction function in (ii) is preferred by Firm A if p_B is smaller than this threshold; otherwise, choosing the reaction function in (iii) is preferred by Firm A. We need to find the threshold value of p_B.

To check the threshold value of p_B for Firm A's reaction function, we derive the profits under cases (ii), (iii), and (iii)'.
The interior profit of firm A under case (ii) for p_B is

```
Factor[pA z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
  z → 3 (pB - pA)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pA →
   ((9 + 3 δc - 10 δf) pB + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
   (2 (9 + 3 δc - 5 δf))}]
```

$$\frac{1}{72\,(a-b)\,t\,(-9+2\delta)}\left(81 a^4 t^2-162 a^2 b^2 t^2+81 b^4 t^2+144 a^2 t^2 \delta+108 a^3 t^2 \delta-18 a^4 t^2 \delta-288 a b t^2 \delta-108 a^2 b t^2 \delta+144 b^2 t^2 \delta-108 a b^2 t^2 \delta+36 a^2 b^2 t^2 \delta+108 b^3 t^2 \delta-18 b^4 t^2 \delta-28 a^2 t^2 \delta^2-12 a^3 t^2 \delta^2+9 a^4 t^2 \delta^2+56 a b t^2 \delta^2+12 a^2 b t^2 \delta^2-28 b^2 t^2 \delta^2+12 a b^2 t^2 \delta^2-18 a^2 b^2 t^2 \delta^2-12 b^3 t^2 \delta^2+9 b^4 t^2 \delta^2-162 a^2 t\, p_B+162 b^2 t\, p_B+36 a t \delta p_B+90 a^2 t \delta p_B-36 b t \delta p_B-90 b^2 t \delta p_B+81 p_B^2\right)$$

The interior profit of firm A under case (iii) for p_B

```
Factor[pA z + δf (1/18 (b - a) t (a + b + 2 z)^2) /. z → (-3 pA + 3 pB - 3 a^2 t + 3 b^2 t + 2 a^2 t δc - 2 b^2 t δc)/(2 (a - b) t (-3 + δc)) /.
  pA → (9 - 3 δc - 2 δf) pB/(2 (9 - 3 δc - δf)) + (b - a) (a + b) t ((3 - δc) (3 - 2 δc) - 2 (2 - δc) δf)/(2 (9 - 3 δc - δf))]
```

$$-\left(\left(-9 a^4 t^2+18 a^2 b^2 t^2-9 b^4 t^2+4 a^4 t^2 \delta-8 a^2 b^2 t^2 \delta+4 b^4 t^2 \delta+18 a^2 t\, p_B-18 b^2 t\, p_B-8 a^2 t \delta p_B+8 b^2 t \delta p_B-9 p_B^2\right) / \left(8\,(a-b)\,t\,(-9+4\delta)\right)\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

```
FullSimplify[
 Solve[{1/(72 (a - b) t (-9 + 2 δ)) (81 a^4 t^2 - 162 a^2 b^2 t^2 + 81 b^4 t^2 + 144 a^2 t^2 δ + 108 a^3 t^2 δ - 18 a^4 t^2 δ -
      288 a b t^2 δ - 108 a^2 b t^2 δ + 144 b^2 t^2 δ - 108 a b^2 t^2 δ + 36 a^2 b^2 t^2 δ + 108 b^3 t^2 δ -
      18 b^4 t^2 δ - 28 a^2 t^2 δ^2 - 12 a^3 t^2 δ^2 + 9 a^4 t^2 δ^2 + 56 a b t^2 δ^2 + 12 a^2 b t^2 δ^2 -
      28 b^2 t^2 δ^2 + 12 a b^2 t^2 δ^2 - 18 a^2 b^2 t^2 δ^2 - 12 b^3 t^2 δ^2 + 9 b^4 t^2 δ^2 - 162 a^2 t pB +
      162 b^2 t pB + 36 a t δ pB + 90 a^2 t δ pB - 36 b t δ pB - 90 b^2 t δ pB + 81 pB^2) ==
    -((-9 a^4 t^2 + 18 a^2 b^2 t^2 - 9 b^4 t^2 + 4 a^4 t^2 δ - 8 a^2 b^2 t^2 δ + 4 b^4 t^2 δ + 18 a^2 t pB -
        18 b^2 t pB - 8 a^2 t δ pB + 8 b^2 t δ pB - 9 pB^2) / (8 (a - b) t (-9 + 4 δ)))}, pB]]
```

$$\left\{\left\{p_B\to-\frac{1}{18\delta}t\,(-9+4\delta)\left(2 a\delta+3a^2\delta-2b\delta-3b^2\delta+3\,(-9+2\delta)\sqrt{\frac{(a-b)^2\,(2+a+b)^2\,\delta^2}{(-9+2\delta)\,(-9+4\delta)}}\right)\right\},\right.$$
$$\left.\left\{p_B\to\frac{1}{18\delta}t\,(-9+4\delta)\left(-2a\delta-3a^2\delta+2b\delta+3b^2\delta+3\,(-9+2\delta)\sqrt{\frac{(a-b)^2\,(2+a+b)^2\,\delta^2}{(-9+2\delta)\,(-9+4\delta)}}\right)\right\}\right\}$$

We can easily show that the former outcome is negative. So, we use the latter one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_B \to \frac{t\,(9-4\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\delta}{9-4\delta}}-(2+3a+3b)\right)}{18} \quad \text{(pb1)}$$

We rewrite the locations of the two endpoints: The right-hand endpoint in (ii), the right-hand endpoints in (iii) (see below)

c2aR
c3aR

$$\left\{\frac{1}{18}(a-b)\,t\,(3a(-3+\delta)+3b(-3+\delta)-2(9+\delta)),\ \frac{1}{9}(-a+b)\,t\,(18+(-2+3a+3b)\,\delta)\right\}$$

$$\left\{\frac{2}{9}(a-b)\,t\,(-9+(5+a+b)\,\delta),\ -\frac{1}{9}(a-b)\,(-4+a+b)\,t\,(-9+4\delta)\right\}$$

We check the condition that the derived p_B (pb1) is below the p_B-element of the right-hand endpoints in (iii).

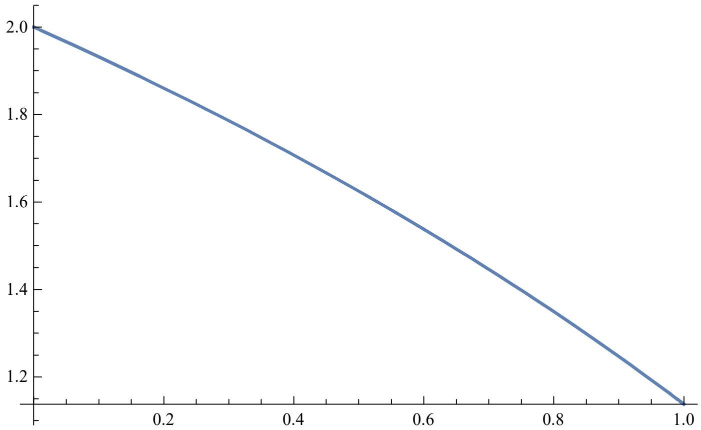
$$\text{Simplify}\left[\text{Factor}\left[-\frac{1}{9}(a-b)\,(-4+a+b)\,t\,(-9+4\delta)-\frac{t\,(9-4\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\delta}{9-4\delta}}-(2+3a+3b)\right)}{18}\right]\right]$$

$$-\frac{1}{18}(a-b)\,t\left(-10+a\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)+b\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)+6\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(-9+4\delta)$$

This is positive if and only if the following inequality holds

$$(a+b)\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)<10-6\sqrt{\frac{9-2\delta}{9-4\delta}}$$

$$\text{Plot}\left[\left(10-6\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\Big/\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right),\ \{\delta,0,1\}\right]$$



If $(a+b)\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)<10-6\sqrt{\frac{9-2\delta}{9-4\delta}}$ holds, the derived p_B (pb1) is on the interval between the left-hand and right-hand endpoints in (iii). Otherwise, the reaction function in (ii) is always better than the interval between the left-hand and right-hand endpoints in (iii).

Now we derive the profit of firm A under case (iii)’ for p_B

$$\text{Factor}\left[p_A\, z + \delta f\left(\frac{1}{18}(b-a)\, t\,(a+b+2z)^2\right) /.\; z \to \frac{-3p_A + 3p_B - 3a^2 t + 3b^2 t + 2a^2 t\,\delta c - 2b^2 t\,\delta c}{2(a-b)\,t\,(-3+\delta c)} /.\; p_A \to p_B + \frac{(b-a)\,t\,((a+b)(3-2\delta c) - 2(3-\delta c))}{3}\right]$$

$$\frac{1}{18}\left(36\,a\,t - 18\,a^2\,t - 36\,b\,t + 18\,b^2\,t - 16\,a\,t\,\delta + 8\,a^2\,t\,\delta - a^3\,t\,\delta + 16\,b\,t\,\delta - a^2\,b\,t\,\delta - 8\,b^2\,t\,\delta + a\,b^2\,t\,\delta + b^3\,t\,\delta + 18\,p_B\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the two profits in cases (ii) and (iii)':

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{\frac{1}{72(a-b)\,t\,(-9+2\delta)}\big(81a^4t^2 - 162a^2b^2t^2 + 81b^4t^2 + 144a^2t^2\delta + 108a^3t^2\delta - 18a^4t^2\delta - 288abt^2\delta - 108a^2bt^2\delta + 144b^2t^2\delta - 108ab^2t^2\delta + 36a^2b^2t^2\delta + 108b^3t^2\delta - 18b^4t^2\delta - 28a^2t^2\delta^2 - 12a^3t^2\delta^2 + 9a^4t^2\delta^2 + 56abt^2\delta^2 + 12a^2bt^2\delta^2 - 28b^2t^2\delta^2 + 12ab^2t^2\delta^2 - 18a^2b^2t^2\delta^2 - 12b^3t^2\delta^2 + 9b^4t^2\delta^2 - 162a^2t\,p_B + 162b^2t\,p_B + 36at\delta\,p_B + 90a^2t\delta\,p_B - 36bt\delta\,p_B - 90b^2t\delta\,p_B + 81p_B^2\big) == \frac{1}{18}\big(36at - 18a^2t - 36bt + 18b^2t - 16at\delta + 8a^2t\delta - a^3t\delta + 16bt\delta - a^2bt\delta - 8b^2t\delta + ab^2t\delta + b^3t\delta + 18p_B\big)\Big\}, p_B\Big]\Big]$$

$$\left\{\left\{p_B \to -\frac{1}{9}(a-b)\,t\left(-9(-4+a+b) + (-6+5a+5b)\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right)\right\}, \left\{p_B \to -\frac{1}{9}(a-b)\,t\left(36 - 9a - 9b - 6\delta + 5(a+b)\,\delta + 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} - 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right)\right\}\right\}$$

We pick up the first outcome as the threshold p_B.

$$p_B \to -\frac{1}{9}(a-b)\,t\left(-9(-4+a+b) + (-6+5a+5b)\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right) \quad \text{(pb2)}$$

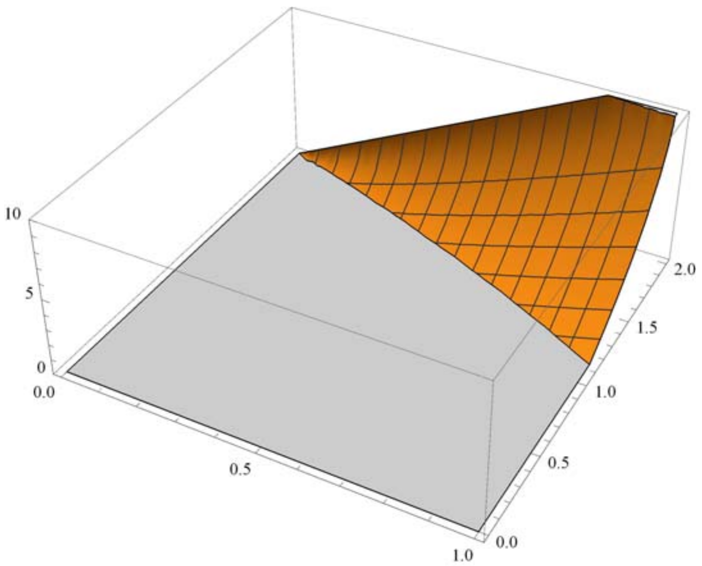
We check the condition that the derived p_B (pb2) is above the p_B-element of the right-hand endpoints in (iii).

$$\text{Simplify}\left[\text{Factor}\left[-\frac{1}{9}(a-b)\,t\left(-9(-4+a+b) + (-6+5a+5b)\,\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right) - \left(-\frac{1}{9}(a-b)(-4+a+b)\,t\,(-9+4\delta)\right)\right]\right]$$

$$-\frac{1}{9}(a-b)\,t\left(10\delta + a\delta + b\delta - 18\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}} + 4\delta\sqrt{\frac{(-2+a+b)(4+a+b)\,\delta}{-9+2\delta}}\right)$$

Denoting a+b by g, we check the value between the largest parentheses:

```
Plot3D[(10 δ + g δ - 18 Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)] + 4 δ Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)]),
 {δ, 0, 1}, {g, 0, 2}, PlotRange → {0, 10}]
```



We check the condition that the value between the large parentheses is positive:

```
Solve[(10 δ + g δ - 18 Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)] + 4 δ Sqrt[((-2 + g) (4 + g) δ)/(-9 + 2 δ)]) == 0, g]
```

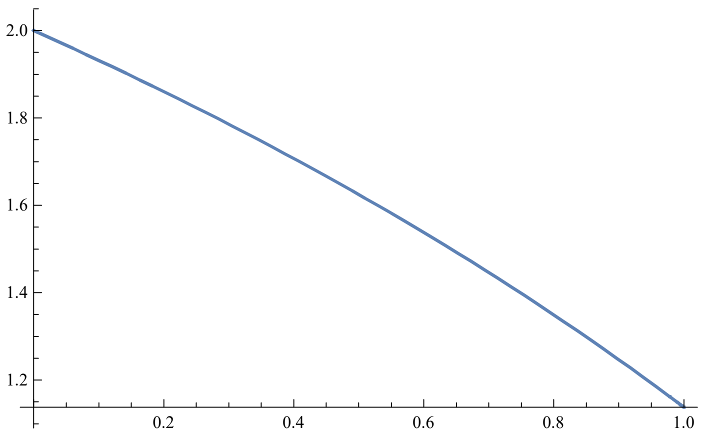
$$\left\{\left\{g \to \frac{2\left(18+\delta-6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}, \left\{g \to \frac{2\left(18+\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}\right\}$$

If $g = a + b > \dfrac{2\left(18+\delta-6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$,

the derived p_B (pb2) is above the p_B-element of the right-hand endpoints in (iii).

This coincides with the following condition that the derived p_B (pb1) is above the p_B-element of the right-hand endpoints in (iii).

```
Plot[(2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), {δ, 0, 1}]
```



If $(a + b) < \dfrac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$ holds, the threshold p_B is on the line segment between the left-hand and right-hand endpoints in (iii). The threshold is given as

$$p_B \to \frac{t\ (9-4\delta)\ (b-a)\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}} - (2+3a+3b)\right)}{18}\ \text{(pb1)}$$

For the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (ii) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A \to \big((9+3\delta c-10\delta f)\ p_B + (b-a)\ t\ \big((a+b)\ (9-\delta c^2-8\delta f+4\delta c\,\delta f) + 2\ (3\delta c+\delta c^2+4\delta f-2\delta c\,\delta f)\big)\big)\Big/ (2\ (9+3\delta c-5\delta f))\ /.\ p_B \to \frac{t\ (9-4\delta)\ (b-a)\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}} - (2+3a+3b)\right)}{18}\Big]\Big]$$

$$p_A \to \frac{1}{36\ (-9+2\delta)}\ (a-b)\ t\ \left(3a\left(-27+81\sqrt{\frac{9-2\delta}{9-4\delta}} + \left(51-99\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta + 2\left(-5+14\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta^2\right) + 3b\left(-27+81\sqrt{\frac{9-2\delta}{9-4\delta}} + \left(51-99\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta + 2\left(-5+14\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta^2\right) + 2\left(81\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right) - 9\left(-25+33\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta + \left(-46+84\sqrt{\frac{9-2\delta}{9-4\delta}}\right)\delta^2\right)\right)$$

We simplify the above outcome, and obtain the following:

$$p_A \to \frac{1}{36\ (9-2\delta)}\ (b-a)\ t\ \left((9-2\delta)\ (-18+46\delta+3\ (a+b)\ (-3+5\delta)) + 3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\ (81-99\delta+28\delta^2)\right)$$

We can define the jumping point of Firm A's reaction function in (ii) as c2ja1

$$\text{c2ja1} = \left\{\frac{1}{36\ (9-2\delta)}\ (b-a)\ t\ \left((9-2\delta)\ (-18+46\delta+3\ (a+b)\ (-3+5\delta)) + 3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\ (81-99\delta+28\delta^2)\right),\ \frac{t\ (9-4\delta)\ (b-a)\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}} - (2+3a+3b)\right)}{18}\right\}$$

$$\left\{\frac{1}{36\ (9-2\delta)}\ (-a+b)\ t\ \left(3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\ (81-99\delta+28\delta^2) + (9-2\delta)\ (-18+46\delta+3\ (a+b)\ (-3+5\delta))\right),\ \frac{1}{18}\ (-a+b)\ t\ \left(-2-3a-3b+3\ (2+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\right)\ (9-4\delta)\right\}$$

Also, for the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (iii) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A \to \frac{(9-3\,\delta c-2\,\delta f)\,p_B}{2\,(9-3\,\delta c-\delta f)}+\frac{(b-a)\,(a+b)\,t\,((3-\delta c)\,(3-2\,\delta c)-2\,(2-\delta c)\,\delta f)}{2\,(9-3\,\delta c-\delta f)}\;/.\;p_B \to \frac{t\,(9-4\,\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(2+3\,a+3\,b)\right)}{18}\Big]\Big]$$

$$p_A \to \frac{1}{36}\,(a-b)\,t\left(2\left(-1+3\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(-9+5\,\delta)+3\,a\left(3-9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}+\delta+5\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\,\delta\right)+3\,b\left(3-9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}+\delta+5\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\,\delta\right)\right)$$

We simplify the above outcome, and obtain the following:

$$p_A \to \frac{1}{36}\,(b-a)\,t\left(-2\,(9-5\,\delta)-3\,(a+b)\,(3+\delta)+3\,(9-5\,\delta)\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)$$

We can define the jumping point of Firm A’s reaction function in (iii) as c3ja1

$$\text{c3ja1} = \left\{\frac{1}{36}\,(b-a)\,t\left(-2\,(9-5\,\delta)-3\,(a+b)\,(3+\delta)+3\,(9-5\,\delta)\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right),\;\frac{t\,(9-4\,\delta)\,(b-a)\left(3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(2+3\,a+3\,b)\right)}{18}\right\}$$

$$\left\{\frac{1}{36}\,(-a+b)\,t\left(-2\,(9-5\,\delta)+3\,(2+a+b)\,(9-5\,\delta)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-3\,(a+b)\,(3+\delta)\right),\;\frac{1}{18}\,(-a+b)\,t\left(-2-3\,a-3\,b+3\,(2+a+b)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta)\right\}$$

If $(a+b) \geq \frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$ holds, the threshold p_B is on the half-line starting from the right-hand endpoints in (iii), that is, case (iii)’. The threshold is given as

$$p_B \to \frac{1}{9}\,(b-a)\,t\left(9\,(4-a-b)-(6-5\,a-5\,b)\,\delta-2\sqrt{(9-2\,\delta)\,(2-a-b)\,(4+a+b)\,\delta}\right)\ \text{(pb2)}$$

For the threshold value of p_B (pb2), the point of p_A-element in Firm A’s reaction function in (ii) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A \to \left((9+3\,\delta c-10\,\delta f)\,p_B+(b-a)\,t\left((a+b)\left(9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f\right)+2\left(3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f\right)\right)\right)/(2\,(9+3\,\delta c-5\,\delta f))\;/.\;p_B \to \frac{1}{9}\,(b-a)\,t\left(9\,(4-a-b)-(6-5\,a-5\,b)\,\delta-2\sqrt{(9-2\,\delta)\,(2-a-b)\,(4+a+b)\,\delta}\right)\Big]\Big]$$

$$p_A \to -\frac{1}{9\,(-9+2\,\delta)}\,(a-b)\,t\left(4\,(-3+a+b)\,\delta^2+9\left(-18+\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}\right)-\delta\left(-90+18\,a+18\,b+7\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}\right)\right)$$

We can define the jumping point of Firm A’s reaction function in (ii) as c2ja2

```
c2ja2 =
```

$$\left\{-\frac{1}{9(-9+2\delta)}(a-b)\,t\left(4(-3+a+b)\delta^2+9\left(-18+\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right)-\delta\left(-90+18a+18b+7\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right)\right),\ \frac{1}{9}(b-a)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(9-2\delta)(2-a-b)(4+a+b)\delta}\right)\right\}$$

$$\left\{-\frac{1}{9(-9+2\delta)}(a-b)\,t\left(4(-3+a+b)\delta^2+9\left(-18+\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right)-\delta\left(-90+18a+18b+7\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right)\right),\ \frac{1}{9}(-a+b)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(2-a-b)(4+a+b)(9-2\delta)\delta}\right)\right\}$$

Also, for the threshold value of p_B (pb2), the point of p_A-element in Firm A's reaction function in (iii)' is

```
Simplify[pA → pB + ((b - a) t ((a + b) (3 - 2 δc) - 2 (3 - δc)))/3 /.
  pB → 1/9 (b - a) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 √((9 - 2 δ) (2 - a - b) (4 + a + b) δ))]
```

$$p_A\to\frac{1}{9}(a-b)\,t\left(-18+a\delta+b\delta+2\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right)$$

We can define the jumping point of Firm A's reaction function in (iii) as c3ja1

```
c3ja2 = {1/9 (a - b) t (-18 + a δ + b δ + 2 √((-8 + a^2 + 2 b + b^2 + 2 a (1 + b)) δ (-9 + 2 δ))),
  1/9 (b - a) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 √((9 - 2 δ) (2 - a - b) (4 + a + b) δ))}
```

$$\left\{\frac{1}{9}(a-b)\,t\left(-18+a\delta+b\delta+2\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right),\ \frac{1}{9}(-a+b)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(2-a-b)(4+a+b)(9-2\delta)\delta}\right)\right\}$$

Next we show various examples of Firm A's true reaction function for different values of a, b, δ, t, and k.

```
a = 0
b = 1
δ = 1/2
t = 1
k = 2
```

0

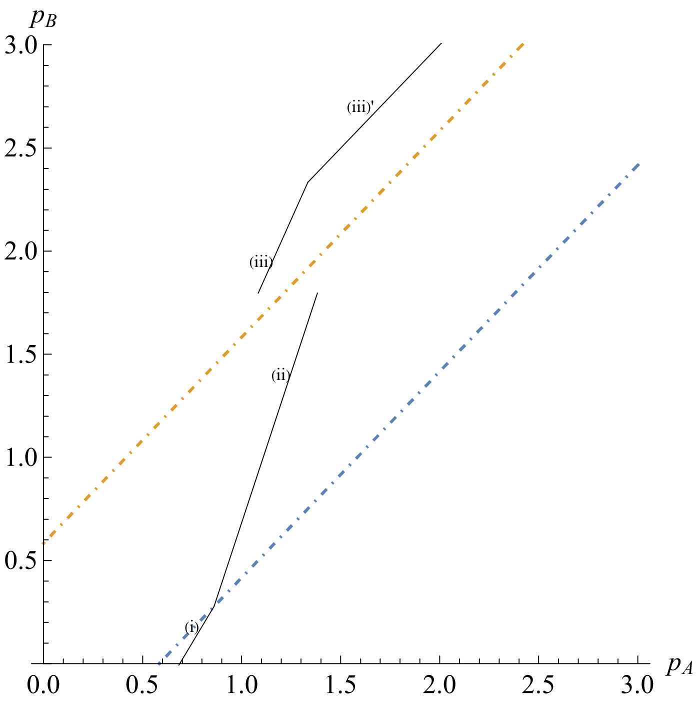
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3}, Epilog → {Line[{c1aL, c1aR}],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text["(iii)'", {1.6, 2.7}], Text["(iii)", {1.1, 1.95}],
   Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0.1376881861101862`
b = 1
δ = 1
t = 1
k = 2
```

0.137688

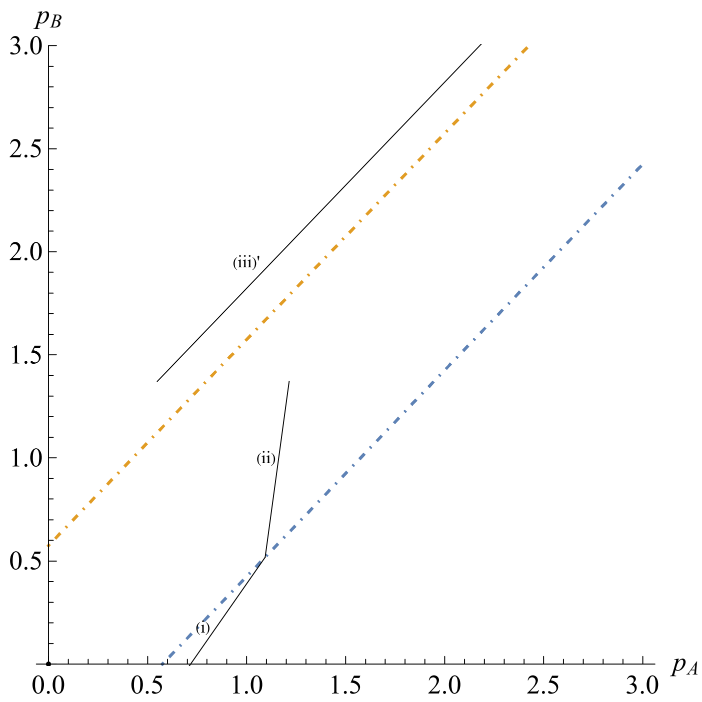
1

1

1

2

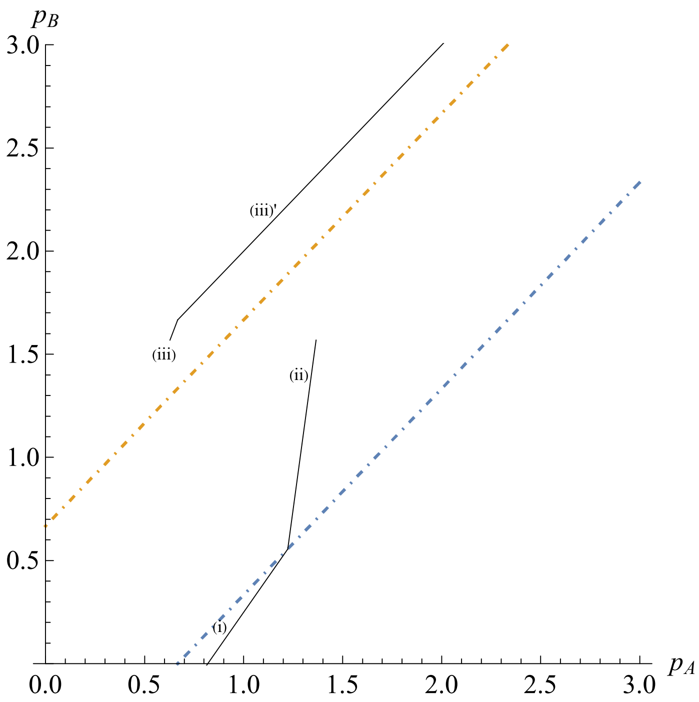
```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3}, Epilog → {Line[{c1aL, c1aR}],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < 2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]) / (36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text["(iii)'", {1, 1.95}], Text["(ii)", {1.1, 1}], Text["(i)", {0.78, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0
b = 1
δ = 1
t = 1
k = 2
```

```
0

1

1

1

2
```

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3}, Epilog → {Line[{c1aL, c1aR}],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text["(iii)'", {1.1, 2.2}], Text["(iii)", {0.6, 1.5}],
   Text["(ii)", {1.28, 1.4}], Text["(i)", {0.88, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```



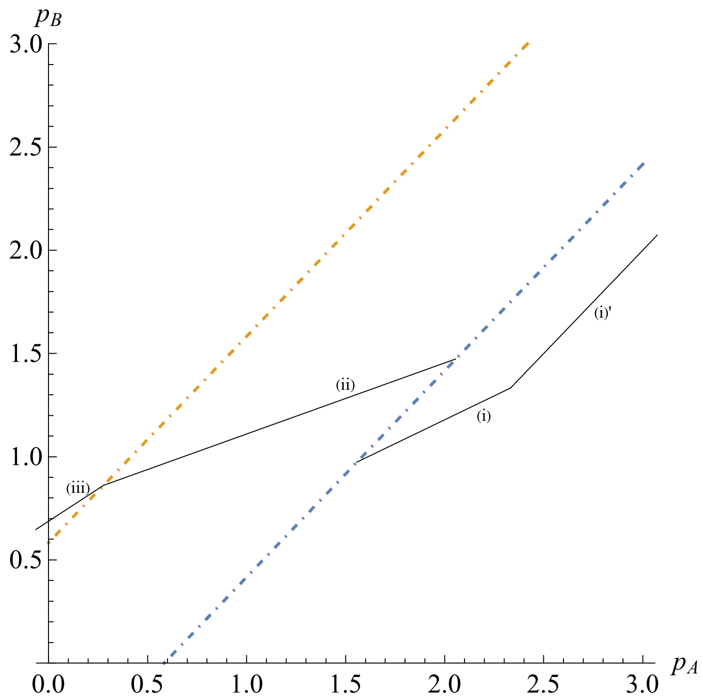

```
Clear[a, b, δ, t, k]
```

We now turn to Firm B's reaction function in the three cases.
As before, we start with an example by setting a = 0, b = 1, t = 1, and $\delta = 1/2$.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 1
```

```
0
1
1/2
1
1
```

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1dbL, c1dbR}], Line[{c1bL, c1bR}], Line[{c2bL, c2bR}],
    Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
    Text["(i)", {2.2, 1.2}], Text["(i)'", {2.8, 1.7}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"p_A", "p_B"}, AspectRatio → 1, PlotStyle → DotDashed]
```



We find that for some p_A, there are two local optimal prices for Firm B. We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the three endpoints: The right-hand endpoint in (ii) (c2bR) and the left-hand and right-hand endpoints in (i) (c1bL and c1bR) (see below)

c2bR
c1bL
c1bR

$$\left\{\frac{1}{9}\,(a-b)\,t\,(-18+(-4+3\,a+3\,b)\,\delta),\ -\frac{1}{18}\,(a-b)\,t\,(-4\,(-9+\delta)+3\,a\,(-3+\delta)+3\,b\,(-3+\delta))\right\}$$

$$\left\{\frac{2}{9}\,(a-b)\,t\,(-9+4\,\delta),\ -\frac{1}{18}\,(a-b)\,t\,(36+9\,a\,(-1+\delta)+9\,b\,(-1+\delta)-28\,\delta)\right\}$$

$$\left\{\frac{1}{9}\,(a-b)\,(2+a+b)\,t\,(-9+4\,\delta),\ -\frac{2}{9}\,(a-b)\,t\,(9+(-7+a+b)\,\delta)\right\}$$

First, we compare the elements of the right-hand endpoint in (ii) and the left-hand endpoint in (i):

$$\text{Factor}\left[\frac{1}{9}\,(a-b)\,t\,(-18+(-4+3\,a+3\,b)\,\delta)-\frac{2}{9}\,(a-b)\,t\,(-9+4\,\delta)\right]$$

$$\text{Factor}\left[-\frac{1}{18}\,(a-b)\,t\,(-4\,(-9+\delta)+3\,a\,(-3+\delta)+3\,b\,(-3+\delta))-\left(-\frac{1}{18}\,(a-b)\,t\,(36+9\,a\,(-1+\delta)+9\,b\,(-1+\delta)-28\,\delta)\right)\right]$$

$$\frac{1}{3}\,(a-b)\,(-4+a+b)\,t\,\delta$$

$$\frac{1}{3}\,(a-b)\,(-4+a+b)\,t\,\delta$$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the right-hand endpoint in (ii) is located above the left-hand endpoint in (i) as in the above Figure.

Second, we compare the p_A-elements of the right-hand endpoint in (ii) and the right-hand endpoint in (i):

$$\text{Factor}\left[\frac{1}{9}\,(a-b)\,t\,(-18+(-4+3\,a+3\,b)\,\delta)-\left(\frac{1}{9}\,(a-b)\,(2+a+b)\,t\,(-9+4\,\delta)\right)\right]$$

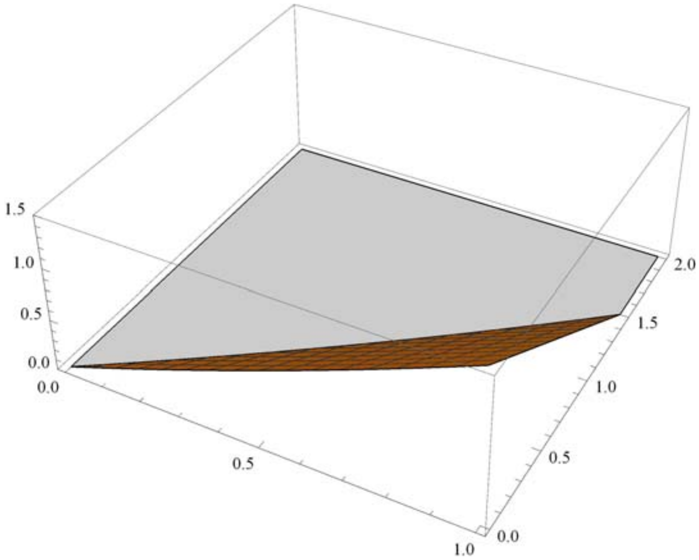
$$-\frac{1}{9}\,(a-b)\,t\,(-9\,a-9\,b+12\,\delta+a\,\delta+b\,\delta)$$

The p_A-element of the right-hand endpoint in (ii) larger than that of the right-hand endpoint in (i) if and only if

$$\frac{12\,\delta}{9-\delta}>a+b$$

If $\frac{12\,\delta}{9-\delta}\le a+b$, we simply compare the reaction function in (ii) and the reaction function in (i); If $\frac{12\,\delta}{9-\delta}>a+b$, in addition to the previous comparison, we also compare the reaction function in (ii) and the reaction function in (i)’.

$$\text{Plot3D}\left[\frac{12\,\delta}{9-\delta} - g,\ \{\delta, 0, 1\},\ \{g, 0, 2\},\ \text{PlotRange} \to \{0, 1.5\}\right]$$



We need to find the global optimal price of Firm B, p_B, when there are two local optima for a given p_A. There is a price p_A such that choosing the reaction function in (i) and choosing the reaction function in (ii) are indifferent for Firm B. This p_A is the threshold in which choosing the reaction function in (i) is preferred by Firm B if p_A is larger than the threshold p_A, otherwise, choosing the reaction function in (ii) is preferred by Firm B. We need to find the threshold value of p_A.

To check the threshold value of p_A for Firm B's reaction function, we derive the profits under cases (i) and (ii).

The interior profit of firm B under case (ii) for p_A is

$$\text{Factor}\Big[p_B\,(1-z) + \delta f\,\frac{1}{9}\,(b-a)\,t\,\left(8 - 4a + a^2 - 4b + 2ab + b^2 - 8z - 2az - 2bz + 10z^2\right)\ /.$$
$$z \to \frac{3\,(p_B - p_A)}{2\,(b-a)\,t\,(3+\delta c)} + \frac{((a+b)\,(3-\delta c) + 2\,\delta c)}{2\,(3+\delta c)}\ /.$$
$$\Big\{p_B \to \frac{1}{2\,(9 + 3\,\delta c - 5\,\delta f)}\,\Big(-(a-b)\,t\,\big(18 + 6\,\delta c - 8\,\delta f + 4\,\delta c\,\delta f + a\,(-9 + \delta c^2 + 8\,\delta f - 4\,\delta c\,\delta f) + b\,(-9 + \delta c^2 + 8\,\delta f - 4\,\delta c\,\delta f)\big) + (9 + 3\,\delta c - 10\,\delta f)\,p_A\Big)\Big\}\Big]$$

$$\frac{1}{72\,(a-b)\,t\,(-9+2\,\delta)}\big(324\,a^2 t^2 - 324\,a^3 t^2 + 81\,a^4 t^2 - 648\,a\,b\,t^2 + 324\,a^2 b\,t^2 + 324\,b^2 t^2 + 324\,a\,b^2 t^2 - 162\,a^2 b^2 t^2 - 324\,b^3 t^2 + 81\,b^4 t^2 + 288\,a^2 t^2 \delta - 36\,a^3 t^2 \delta - 18\,a^4 t^2 \delta - 576\,a\,b\,t^2 \delta + 36\,a^2 b\,t^2 \delta + 288\,b^2 t^2 \delta + 36\,a\,b^2 t^2 \delta + 36\,a^2 b^2 t^2 \delta - 36\,b^3 t^2 \delta - 18\,b^4 t^2 \delta - 16\,a^2 t^2 \delta^2 - 24\,a^3 t^2 \delta^2 + 9\,a^4 t^2 \delta^2 + 32\,a\,b\,t^2 \delta^2 + 24\,a^2 b\,t^2 \delta^2 - 16\,b^2 t^2 \delta^2 + 24\,a\,b^2 t^2 \delta^2 - 18\,a^2 b^2 t^2 \delta^2 - 24\,b^3 t^2 \delta^2 + 9\,b^4 t^2 \delta^2 - 324\,a\,t\,p_A + 162\,a^2 t\,p_A + 324\,b\,t\,p_A - 162\,b^2 t\,p_A + 216\,a\,t\,\delta\,p_A - 90\,a^2 t\,\delta\,p_A - 216\,b\,t\,\delta\,p_A + 90\,b^2 t\,\delta\,p_A + 81\,p_A^2\big)$$

The interior profit of firm B under case (i) for p_B

$$\text{Factor}\Big[p_B\,(1-z)+\delta f\left(\frac{1}{18}\,(b-a)\,t\,(-4+a+b+2z)^2\right)\;/.\;z\to\left(-3a^2t+3b^2t-2at\delta c+2a^2t\delta c+2bt\delta c-2b^2t\delta c-3p_A+3p_B\right)/\left(2\,(a-b)\,t\,(-3+\delta c)\right)\;/.\;p_B\to\left(-\left(-2a+a^2-(-2+b)\,b\right)t\left(9+2\delta c^2-4\delta f+\delta c\,(-9+2\delta f)\right)+(-9+3\delta c+2\delta f)\,p_A\right)/\left(2\,(-9+3\delta c+\delta f)\right)\Big]$$

$$-\frac{1}{8\,(a-b)\,t\,(-9+4\delta)}\left(-36a^2t^2+36a^3t^2-9a^4t^2+72abt^2-36a^2bt^2-36b^2t^2-36ab^2t^2+18a^2b^2t^2+36b^3t^2-9b^4t^2+16a^2t^2\delta-16a^3t^2\delta+4a^4t^2\delta-32abt^2\delta+16a^2bt^2\delta+16b^2t^2\delta+16ab^2t^2\delta-8a^2b^2t^2\delta-16b^3t^2\delta+4b^4t^2\delta+36atp_A-18a^2tp_A-36btp_A+18b^2tp_A-16at\delta p_A+8a^2t\delta p_A+16bt\delta p_A-8b^2t\delta p_A-9p_A^2\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{\frac{1}{72\,(a-b)\,t\,(-9+2\delta)}\left(324a^2t^2-324a^3t^2+81a^4t^2-648abt^2+324a^2bt^2+324b^2t^2+324ab^2t^2-162a^2b^2t^2-324b^3t^2+81b^4t^2+288a^2t^2\delta-36a^3t^2\delta-18a^4t^2\delta-576abt^2\delta+36a^2bt^2\delta+288b^2t^2\delta+36ab^2t^2\delta+36a^2b^2t^2\delta-36b^3t^2\delta-18b^4t^2\delta-16a^2t^2\delta^2-24a^3t^2\delta^2+9a^4t^2\delta^2+32abt^2\delta^2+24a^2bt^2\delta^2-16b^2t^2\delta^2+24ab^2t^2\delta^2-18a^2b^2t^2\delta^2-24b^3t^2\delta^2+9b^4t^2\delta^2-324atp_A+162a^2tp_A+324btp_A-162b^2tp_A+216at\delta p_A-90a^2t\delta p_A-216bt\delta p_A+90b^2t\delta p_A+81p_A^2\right)==-\frac{1}{8\,(a-b)\,t\,(-9+4\delta)}\left(-36a^2t^2+36a^3t^2-9a^4t^2+72abt^2-36a^2bt^2-36b^2t^2-36ab^2t^2+18a^2b^2t^2+36b^3t^2-9b^4t^2+16a^2t^2\delta-16a^3t^2\delta+4a^4t^2\delta-32abt^2\delta+16a^2bt^2\delta+16b^2t^2\delta+16ab^2t^2\delta-8a^2b^2t^2\delta-16b^3t^2\delta+4b^4t^2\delta+36atp_A-18a^2tp_A-36btp_A+18b^2tp_A-16at\delta p_A+8a^2t\delta p_A+16bt\delta p_A-8b^2t\delta p_A-9p_A^2\right)\Big\},\,p_A\Big]\Big]$$

$$\left\{\left\{p_A\to-\frac{1}{18\,\delta}t\,(-9+4\delta)\left(8a\delta-3a^2\delta-8b\delta+3b^2\delta+3\,(-9+2\delta)\sqrt{\frac{(a-b)^2\,(-4+a+b)^2\,\delta^2}{(-9+2\delta)\,(-9+4\delta)}}\right)\right\},\left\{p_A\to\frac{1}{18\,\delta}t\,(-9+4\delta)\left(-8a\delta+3a^2\delta+8b\delta-3b^2\delta+3\,(-9+2\delta)\sqrt{\frac{(a-b)^2\,(-4+a+b)^2\,\delta^2}{(-9+2\delta)\,(-9+4\delta)}}\right)\right\}\right\}$$

We can easily show that the former outcome is negative. So, we use the latter one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_A\to\frac{t\,(9-4\delta)\,(b-a)\left(3\,(4-a-b)\sqrt{\frac{9-2\delta}{9-4\delta}}-(8-3a-3b)\right)}{18}\quad(\text{pa1})$$

We rewrite the locations of the two endpoints: The right-hand endpoint in (ii) and the right-hand endpoint in (i) (see below)

```
c2bR
c1bR
```

$$\left\{\frac{1}{9}\,(a-b)\,t\,(-18+(-4+3a+3b)\,\delta),\;-\frac{1}{18}\,(a-b)\,t\,(-4\,(-9+\delta)+3a\,(-3+\delta)+3b\,(-3+\delta))\right\}$$

$$\left\{\frac{1}{9}\,(a-b)\,(2+a+b)\,t\,(-9+4\delta),\;-\frac{2}{9}\,(a-b)\,t\,(9+(-7+a+b)\,\delta)\right\}$$

We check the condition that the derived p_A (pa1) is smaller than the p_A-element of the right-hand endpoints in (i).

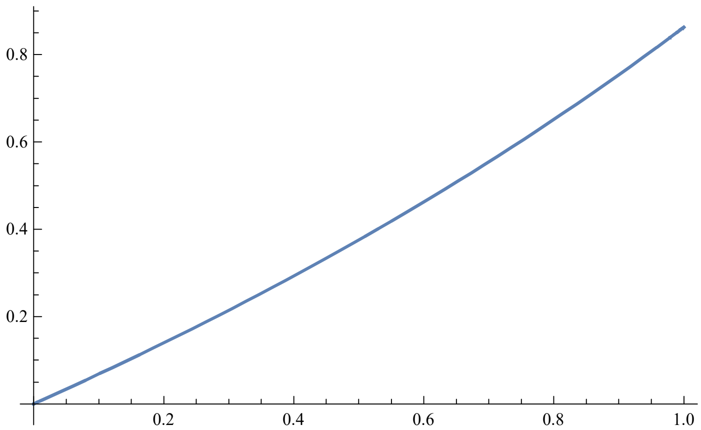
$$\text{Simplify}\left[\text{Factor}\left[\frac{1}{9}(a-b)(2+a+b)\,t\,(-9+4\delta)-\frac{t\,(9-4\delta)\,(b-a)\left(3\,(4-a-b)\sqrt{\frac{9-2\delta}{9-4\delta}}-(8-3a-3b)\right)}{18}\right]\right]$$

$$\frac{1}{18}(a-b)\,t\left(12-b+a\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)-12\sqrt{\frac{9-2\delta}{9-4\delta}}+3b\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(-9+4\delta)$$

This is positive if the following inequality holds

$$(a+b)\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)>12\left(\sqrt{\frac{9-2\delta}{9-4\delta}}-1\right)$$

$$\text{Plot}\left[12\left(\sqrt{\frac{9-2\delta}{9-4\delta}}-1\right)\Big/\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right),\{\delta,0,1\}\right]$$



If $(a+b)\left(-1+3\sqrt{\frac{9-2\delta}{9-4\delta}}\right)>12\left(\sqrt{\frac{9-2\delta}{9-4\delta}}-1\right)$ holds, the derived p_A (pa1) is on the interval between the left-hand and right-hand endpoints in (i). Otherwise, the reaction function in (ii) is always better than the interval between the left-hand and right-hand endpoints in (i).

Now we derive the profit of firm B under case (i)' for p_A

$$\text{Factor}\left[p_B\,(1-z)+\delta f\left(\frac{1}{18}(b-a)\,t\,(-4+a+b+2z)^2\right)\;/.\right.$$
$$z\to\left(-3a^2t+3b^2t-2at\,\delta c+2a^2t\,\delta c+2bt\,\delta c-2b^2t\,\delta c-3p_A+3p_B\right)\big/\left(2\,(a-b)\,t\,(-3+\delta c)\right)\;/.$$
$$\left.p_B\to p_A-\frac{(b-a)\,t\,((a+b)(3-2\delta c)+2\delta c)}{3}\right]$$

$$\frac{1}{18}\left(18a^2t-18b^2t-4at\delta-4a^2t\delta-a^3t\delta+4bt\delta-a^2bt\delta+4b^2t\delta+ab^2t\delta+b^3t\delta+18p_A\right)$$

We derive the threshold value of p_A by finding p_A that equalizes the two profits in cases (ii) and (i)':

```
FullSimplify[
 Solve[{1/(72 (a - b) t (-9 + 2 δ)) (324 a^2 t^2 - 324 a^3 t^2 + 81 a^4 t^2 - 648 a b t^2 + 324 a^2 b t^2 + 324 b^2 t^2 +
      324 a b^2 t^2 - 162 a^2 b^2 t^2 - 324 b^3 t^2 + 81 b^4 t^2 + 288 a^2 t^2 δ - 36 a^3 t^2 δ - 18 a^4 t^2 δ -
      576 a b t^2 δ + 36 a^2 b t^2 δ + 288 b^2 t^2 δ + 36 a b^2 t^2 δ + 36 a^2 b^2 t^2 δ - 36 b^3 t^2 δ - 18 b^4 t^2 δ -
      16 a^2 t^2 δ^2 - 24 a^3 t^2 δ^2 + 9 a^4 t^2 δ^2 + 32 a b t^2 δ^2 + 24 a^2 b t^2 δ^2 - 16 b^2 t^2 δ^2 + 24 a b^2 t^2 δ^2 -
      18 a^2 b^2 t^2 δ^2 - 24 b^3 t^2 δ^2 + 9 b^4 t^2 δ^2 - 324 a t p_A + 162 a^2 t p_A + 324 b t p_A - 162 b^2 t p_A +
      216 a t δ p_A - 90 a^2 t δ p_A - 216 b t δ p_A + 90 b^2 t δ p_A + 81 p_A^2) == 1/18 (18 a^2 t - 18 b^2 t -
      4 a t δ - 4 a^2 t δ - a^3 t δ + 4 b t δ - a^2 b t δ + 4 b^2 t δ + a b^2 t δ + b^3 t δ + 18 p_A)}, p_A]]
```

$$\left\{\left\{p_A \to \frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\},\right.$$
$$\left.\left\{p_A \to \frac{1}{9}(a-b)\,t\left(-9(2+a+b)+(-4+5a+5b)\delta-18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}+4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}\right\}$$

We pick up the first outcome as the threshold p_A.

$$p_A \to \frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right) \quad \text{(pa2)}$$

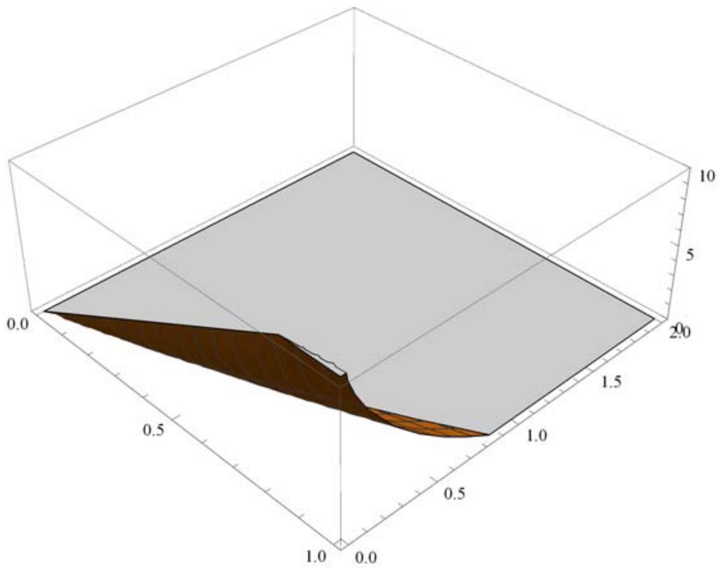
We check the condition that the derived p_A (pa2) is above the p_A-element of the right-hand endpoints in (i).

```
Simplify[Factor[1/9 (a - b) t
    (-9 a - 9 (2 + b) - 4 δ + 5 (a + b) δ + 18 Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)]) -
   1/9 (a - b) (2 + a + b) t (-9 + 4 δ)]]
```

$$\frac{1}{9}(a-b)\,t\left(-12\delta+a\delta+b\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)$$

By setting a+b by g, we check the value between the large parentheses:

```
Plot3D[-(-12 δ + g δ + 18 Sqrt[((-6 + g) (g) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + g) (g) δ)/(-9 + 2 δ)]),
 {δ, 0, 1}, {g, 0, 2}, PlotRange → {0, 10}]
```



We check the condition that the value between the large parentheses is negative:

```
Solve[(-12 δ + g δ + 18 Sqrt[((-6 + g) (g) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + g) (g) δ)/(-9 + 2 δ)]) == 0, g]
```

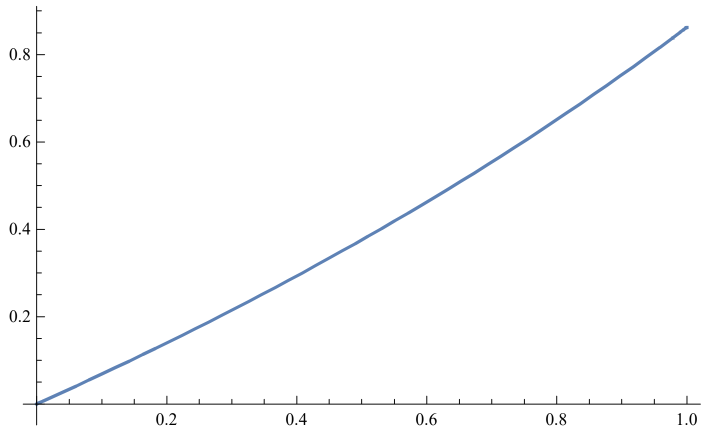
$$\left\{\left\{g \to \frac{12\left(-9+\delta-\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}, \left\{g \to \frac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}\right\}\right\}$$

If $g = a + b < \dfrac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$,

the derived p_A (pa2) is above the p_A-element of the right-hand endpoints in (i).

This coincides with the following condition that the derived p_A (pa1) is above the p_A-element of the right-hand endpoints in (i).

```
Plot[(12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), {δ, 0, 1}]
```



If $(a + b) > \dfrac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$ holds, the threshold p_B is on the line segment between the left-hand and right-hand endpoints in (i). The threshold is given as

$$p_A \to \frac{t\ (9-4\delta)\ (b-a)\ \left(3\ (4-a-b)\ \sqrt{\frac{9-2\delta}{9-4\delta}} - (8-3a-3b)\right)}{18}$$

The reaction function of Firm B in (i) is

$$p_B \to \frac{1}{2\ (9-3\delta c-\delta f)}\left(((b-a)\ (2-a-b))\ t\ \left(9+2\delta c^2-4\delta f+\delta c\ (-9+2\delta f)\right)+(9-3\delta c-2\delta f)\ p_A\right)$$

We substitute p_A into p_B:

```
Simplify[pB →
   (((b - a) (2 - a - b)) t (9 + 2 δc^2 - 4 δf + δc (-9 + 2 δf)) + (9 - 3 δc - 2 δf) pA) / (2 (9 - 3 δc - δf)) /.
   pA → t (9 - 4 δ) (b - a) (3 (4 - a - b) Sqrt[(9-2 δ)/(9-4 δ)] - (8 - 3 a - 3 b)) / 18]
```

$$p_B \to \frac{1}{36\ (9-4\delta)}\ (-a+b)\ t\ \left(\left(-8+3a+3b-3\ (-4+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\right)\ (9-5\delta)\ (9-4\delta)-18\ (-2+a+b)\ \left(9-13\delta+4\delta^2\right)\right)$$

We can define the jumping point of Firm B's reaction function in (i) as c1jb1

$$\text{c1jb1} = \left\{\frac{t\ (9-4\delta)\ (b-a)\ \left(3\ (4-a-b)\ \sqrt{\frac{9-2\delta}{9-4\delta}} - (8-3a-3b)\right)}{18},\ \frac{1}{36\ (9-4\delta)}\ (-a+b)\ t\ \left(\left(-8+3a+3b-3\ (-4+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\right)\ (9-5\delta)\ (9-4\delta)-18\ (-2+a+b)\ \left(9-13\delta+4\delta^2\right)\right)\right\}$$

$$\left\{\frac{1}{18}\ (-a+b)\ t\ \left(-8+3a+3b+3\ (4-a-b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\right)\ (9-4\delta),\ \frac{1}{36\ (9-4\delta)}\ (-a+b)\ t\ \left(\left(-8+3a+3b-3\ (-4+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\right)\ (9-5\delta)\ (9-4\delta)-18\ (-2+a+b)\ \left(9-13\delta+4\delta^2\right)\right)\right\}$$

The reaction function of Firm B in (ii) is

$$p_B \to \frac{1}{2\ (9+3\delta c-5\delta f)}\left(-(a-b)\ t\ \left(18+6\delta c-8\delta f+4\delta c\,\delta f+a\left(-9+\delta c^2+8\delta f-4\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\delta f-4\delta c\,\delta f\right)\right)+(9+3\delta c-10\delta f)\ p_A\right)$$

We substitute p_A into p_B:

```
Simplify[pB → 1/(2 (9 + 3 δc - 5 δf))
   (-(a - b) t (18 + 6 δc - 8 δf + 4 δc δf + a (-9 + δc^2 + 8 δf - 4 δc δf) + b (-9 + δc^2 + 8 δf - 4 δc δf)) +
     (9 + 3 δc - 10 δf) pA) /. pA → t (9 - 4 δ) (b - a) (3 (4 - a - b) Sqrt[(9-2 δ)/(9-4 δ)] - (8 - 3 a - 3 b)) / 18]
```

$$p_B \to \frac{1}{126}\ (-a+b)\ t\ \left(-36\ (-5+a+b)+\left(-8+3a+3b-3\ (-4+a+b)\ \sqrt{\frac{9-2\delta}{9-4\delta}}\right)\ (9-4\delta)\right)$$

We can define the jumping point of Firm B's reaction function in (ii) as c2jb1

$$\text{c2jb1} = \left\{\frac{t\,(9-4\,\delta)\,(b-a)\left(3\,(4-a-b)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}} - (8-3\,a-3\,b)\right)}{18},\right.$$
$$\left.\frac{1}{126}\,(-a+b)\,t\left(-36\,(-5+a+b) + \left(-8+3\,a+3\,b-3\,(-4+a+b)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta)\right)\right\}$$

$$\left\{\frac{1}{18}\,(-a+b)\,t\left(-8+3\,a+3\,b+3\,(4-a-b)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta),\right.$$
$$\left.\frac{1}{126}\,(-a+b)\,t\left(-36\,(-5+a+b) + \left(-8+3\,a+3\,b-3\,(-4+a+b)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta)\right)\right\}$$

If $(a+b) \le \frac{12\left(-9+\delta+\sqrt{81-54\,\delta+8\,\delta^2}\right)}{-36+7\,\delta}$ holds, the threshold p_A is on the half-line starting from the right-hand endpoints in (i), that is, case (i)'. The threshold is given as

$$p_A \to \frac{1}{9}\,(a-b)\,t\left(-9\,a-9\,(2+b)-4\,\delta+5\,(a+b)\,\delta+18\,\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} - 4\,\delta\,\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right)$$

The reaction functions of Firm B in (i)' and (ii) are

$$p_B \to p_A - \frac{(b-a)\,t\,((a+b)\,(3-2\,\delta c)+2\,\delta c)}{3}$$

$$p_B \to \left((b-a)\,t\,\left(2\,(9+3\,\delta c-4\,\delta f+2\,\delta c\,\delta f) + (a+b)\,\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right) + (9+3\,\delta c-10\,\delta f)\,p_A\right) \Big/ (2\,(9+3\,\delta c-5\,\delta f))$$

We substitute p_A into p_B:

```
Simplify[pB → pA - ((b - a) t ((a + b) (3 - 2 δc) + 2 δc))/3 /. pA → 1/9 (a - b) t
   (-9 a - 9 (2 + b) - 4 δ + 5 (a + b) δ + 18 Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)])]
Simplify[pB → ((b - a) t (2 (9 + 3 δc - 4 δf + 2 δc δf) + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf)) +
      (9 + 3 δc - 10 δf) pA) / (2 (9 + 3 δc - 5 δf)) /. pA → 1/9 (a - b) t
   (-9 a - 9 (2 + b) - 4 δ + 5 (a + b) δ + 18 Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)] - 4 δ Sqrt[((-6 + a + b) (a + b) δ)/(-9 + 2 δ)])]
```

$$p_B \to -\frac{1}{9}\,(a-b)\,t\left(18-2\,\delta+a\,\delta+b\,\delta-18\,\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} + 4\,\delta\,\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right)$$

$$p_B \to \frac{1}{9}\,(a-b)\,t\left(-18+2\,\delta+2\,a\,\delta+2\,b\,\delta+9\,\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}} - 7\,\delta\,\sqrt{\frac{(-6+a+b)\,(a+b)\,\delta}{-9+2\,\delta}}\right)$$

We can define the jumping point of Firm B's reaction function in (i) as c1jb2

c1jb2 =

$$\left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\right.$$
$$\left.-\frac{1}{9}(a-b)\,t\left(18-2\delta+a\delta+b\delta-18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}+4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\right.$$
$$\left.-\frac{1}{9}(a-b)\,t\left(18-2\delta+a\delta+b\delta-18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}+4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

Also, we can define the jumping point of Firm B's reaction function in (ii) as c2jb2

c2jb2 =

$$\left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\right.$$
$$\left.\frac{1}{9}(a-b)\,t\left(-18+2\delta+2a\delta+2b\delta+9\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-7\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\left(-9a-9(2+b)-4\delta+5(a+b)\delta+18\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-4\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right),\right.$$
$$\left.\frac{1}{9}(a-b)\,t\left(-18+2\delta+2a\delta+2b\delta+9\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}-7\delta\sqrt{\frac{(-6+a+b)(a+b)\delta}{-9+2\delta}}\right)\right\}$$

As before, we now show various examples of Firm B's true reaction function for different values of a, b, δ, t, and k.

```
a = 0
b = 1
δ = 1/2
t = 1
k = 2
```

0

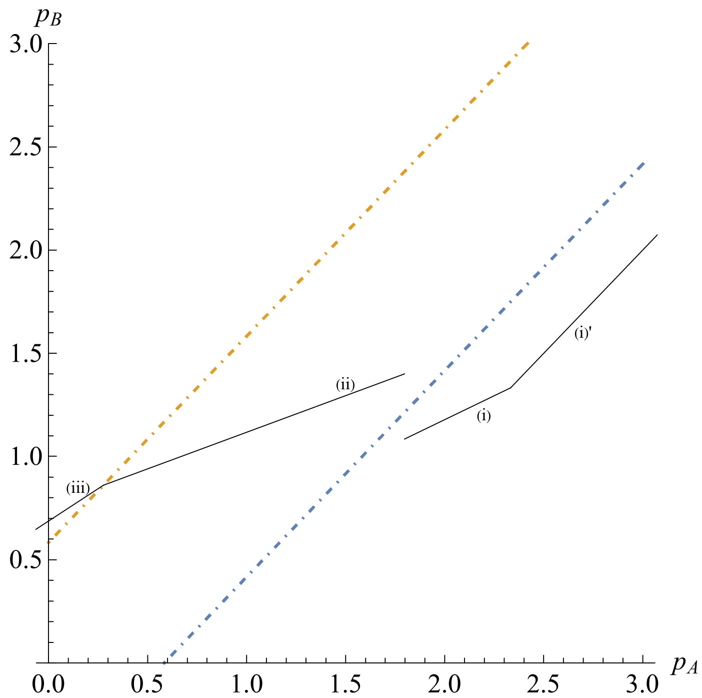
1

$\frac{1}{2}$

1

2

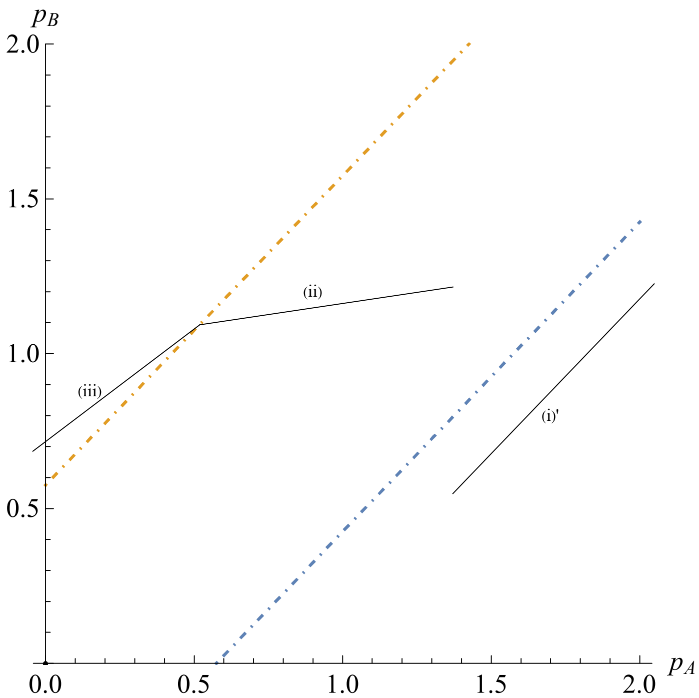
```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
 Epilog → {If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c1dbL, c1dbR}], Line[{c1jb2, c1dbR}]],
   If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c1jb1, c1bR}], Line[{{0, 0}, {0, 0}}] ],
   If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c2bL, c2jb1}], Line[{c2bL, c2jb2}]],
   Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
   Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0
b = 1 - 0.1376881861101862`
δ = 1
t = 1
k = 2
```

```
0

0.862312

1

1

2
```

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 2},
 Epilog → {If[a + b > 12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]) / (-36 + 7 δ), Line[{c1dbL, c1dbR}], Line[{c1jb2, c1dbR}]],
   If[a + b > 12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]) / (-36 + 7 δ), Line[{c1jb1, c1bR}], Line[{{0, 0}, {0, 0}}] ],
   If[a + b > 12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]) / (-36 + 7 δ), Line[{c2bL, c2jb1}], Line[{c2bL, c2jb2}]],
   Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.88}], Text["(ii)", {0.9, 1.2}],
   Text["(i)'", {1.7, 0.8}]}, PlotRange → {0, 2}, LabelStyle → (FontSize → 14),
 AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```

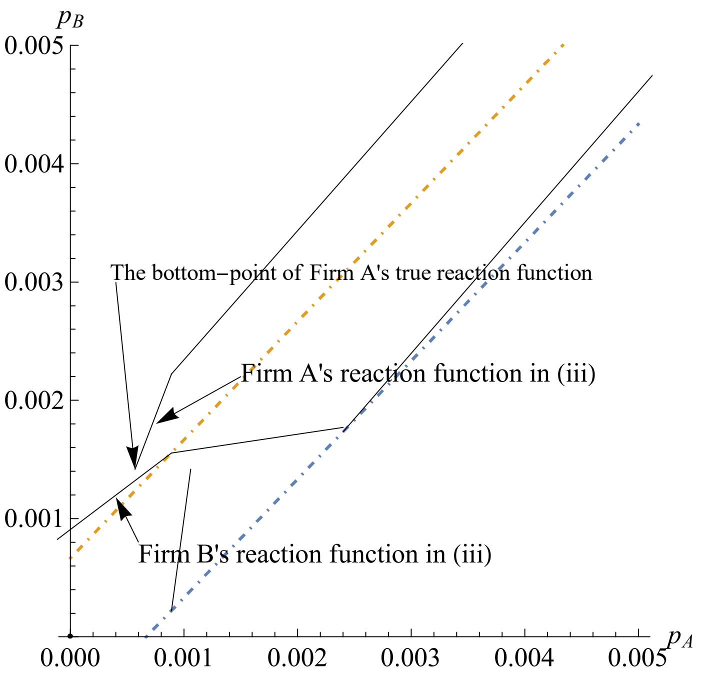


We now show an example in which there is no intersection between the two firms' reaction functions.

```
a = 0
b = 0.001
δ = 1
t = 1
k = 2
```

```
0

0.001

1

1

2
```

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 0.005},
 Epilog → {If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c1dbL, c1dbR}], Line[{c1jb2, c1dbR}]],
   If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c1jb1, c1bR}], Line[{{0, 0}, {0, 0}}] ],
   If[a + b > (12 (-9 + δ + Sqrt[81 - 54 δ + 8 δ^2]))/(-36 + 7 δ), Line[{c2bL, c2jb1}], Line[{c2bL, c2jb2}]],
   Line[{c3bL, c3bR}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
   Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}], , , , Line[{c1aL, c1aR}],
   If[ a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c2aL, c2ja1}], Line[{c2aL, c2ja2}]],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c3ja1, c3aR}], Line[{{0, 0}, {0, 0}}]],
   If[a + b < (2 (-18 - δ + 6 Sqrt[81 - 54 δ + 8 δ^2]))/(36 - 7 δ), Line[{c3daL, c3daR}], Line[{c3ja2, c3daR}]],
   Text[Style["Firm A's reaction function in (iii)", FontSize → 14],
    {0.0015, 0.0021}, {-1, -1}], Arrow[{{0.0004, 0.003}, c3ja1}],
   Arrow[{{0.0015, 0.0022}, {0.00075, 0.0018}}], Arrow[{{0.0006, 0.0008}, {0.0004, 0.00118}}],
   Text[Style["The bottom-point of Firm A's true reaction function", FontSize → 12],
    {0.00035, 0.00308}, {-1, 0}],
   Text[Style["Firm B's reaction function in (iii)", FontSize → 14], {0.0006, 0.00083},
    {-1, 1}], Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
 PlotRange → {0, 0.005}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```



We now turn to the pricing equilibrium in the first period. First, we show that no equilibrium exists in (Case iii).
To this end, we show that the lowest point of Firm A's reaction function in (iii) is above Firm B's reaction function in (iii), as shown in the previous figure. If this property holds, Firm A's reaction function in (iii) never passes through the reaction function of Firm B in (iii) because the slope of Firm A's reaction function in (iii) is steeper than that of Firm B in (iii).

```
Clear[a, b, δ, t, k]
δf = δ
δc = δ
```

δ

δ

The lowest point of Firm A's "true" reaction function in (iii) is point "c3ja1" if if $a + b < \frac{2\left(-18-\delta+6\sqrt{81-54\delta+8\delta^2}\right)}{36-7\delta}$, otherwise, point "c3ja2". We show the two points in the following:

```
c3ja1
c3ja2
```

$$\left\{\frac{1}{36}(-a+b)\,t\left(-2(9-5\delta)+3(2+a+b)(9-5\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}-3(a+b)(3+\delta)\right),\right.$$
$$\left.\frac{1}{18}(-a+b)\,t\left(-2-3a-3b+3(2+a+b)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)(9-4\delta)\right\}$$

$$\left\{\frac{1}{9}(a-b)\,t\left(-18+a\delta+b\delta+2\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right),\right.$$
$$\left.\frac{1}{9}(-a+b)\,t\left(9(4-a-b)-(6-5a-5b)\delta-2\sqrt{(2-a-b)(4+a+b)(9-2\delta)\delta}\right)\right\}$$

The following is the reaction function of Firm B in (iii).

$$pb\to\frac{(9-3\,\delta c+10\,\delta f)\,pa}{2\,(9-3\,\delta c+5\,\delta f)}+$$
$$((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/$$
$$(2\,(9-3\,\delta c+5\,\delta f))$$

We substitute p_A-element of point "c3ja1" into the reaction function of Firm B in (iii).

```
Simplify[pb → (9 - 3 δc + 10 δf) pa / (2 (9 - 3 δc + 5 δf)) +
   ((b - a) t (2 (3 - δc) (3 - δc + 3 δf) - (a + b) ((3 - δc) (3 - 2 δc) + 5 (1 - δc) δf))) /
    (2 (9 - 3 δc + 5 δf)) /. pa → 1/36 (-a + b) t
   (-2 (9 - 5 δ) + 3 (2 + a + b) (9 - 5 δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - 3 (a + b) (3 + δ)) /. {δf → δ, δc → δ}]
```

$$pb\to\frac{1}{72(9+2\delta)}(-a+b)\,t\left((9+7\delta)\left(3(2+a+b)(9-5\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}-3(a+b)(3+\delta)+2(-9+5\delta)\right)+\right.$$
$$\left.36\left(18+6\delta-4\delta^2+a\left(-9+4\delta+3\delta^2\right)+b\left(-9+4\delta+3\delta^2\right)\right)\right)$$

We check if this derived value of p_B is smaller than p_B-element of point "c3ja1". If the derived value of p_B is actually smaller than p_B-element of point "c3ja1", Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii).
The difference between the derived value of p_B and the p_B-element of c3ja1:

```
Simplify[Factor[
  1/(72 (9 + 2 δ)) (-a + b) t ((9 + 7 δ) (3 (2 + a + b) (9 - 5 δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - 3 (a + b) (3 + δ) + 2 (-9 + 5 δ)) +
     36 (18 + 6 δ - 4 δ^2 + a (-9 + 4 δ + 3 δ^2) + b (-9 + 4 δ + 3 δ^2))) -
  1/18 (-a + b) t (-2 - 3 a - 3 b + 3 (2 + a + b) Sqrt[(9 - 2 δ)/(9 - 4 δ)]) (9 - 4 δ)]]
```

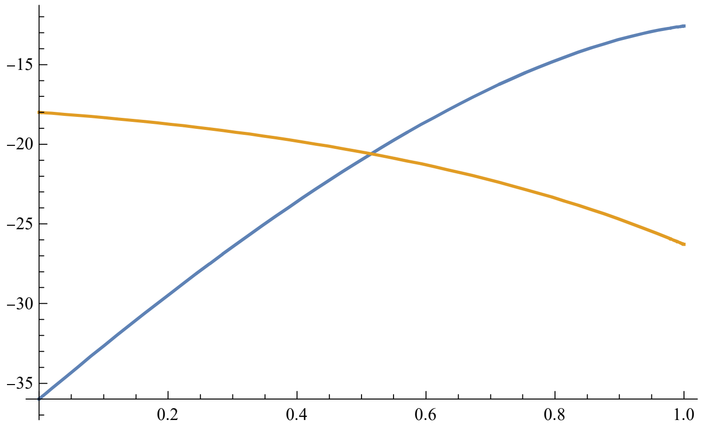
$$\frac{1}{24\,(9+2\delta)}(a-b)\,t\,(-3+\delta)\left(3a\left(21-27\sqrt{\frac{9-2\delta}{9-4\delta}}+\delta+\sqrt{\frac{9-2\delta}{9-4\delta}}\,\delta\right)+3b\left(21+\delta-27\sqrt{\frac{-9+2\delta}{-9+4\delta}}+\delta\sqrt{\frac{-9+2\delta}{-9+4\delta}}\right)+2\left(63+23\delta-81\sqrt{\frac{-9+2\delta}{-9+4\delta}}+3\delta\sqrt{\frac{-9+2\delta}{-9+4\delta}}\right)\right)$$

We rearrange this value, and obtain the following:

$$\frac{1}{24\,(9+2\delta)}(b-a)\,t\,(3-\delta)\left(\left((2\,(63+23\delta))-6\,(27-\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}\right)-3\,(a+b)\left((27-\delta)\sqrt{\frac{9-2\delta}{9-4\delta}}-(21+\delta)\right)\right)$$

We check the values of the first and the second terms in the largest parentheses

```
Plot[{(2 (63 + 23 δ)) - 6 (27 - δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)], -3 ((27 - δ) Sqrt[(9 - 2 δ)/(9 - 4 δ)] - (21 + δ))}, {δ, 0, 1}]
```



Both values are negative. Therefore, the difference between the derived value of p_B and the p_B-element of c3ja1 is negative. That is, Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii) `if` $a + b < \frac{2\left(-18-\delta+6\sqrt{81-54\delta+8\delta^2}\right)}{36-7\delta}$.

We substitute p_A-element of c3ja2 into the reaction function of Firm B in (iii).

```
Simplify[
 pb → 1/9 (a - b) t (-18 + a δ + b δ + 2 √((-8 + a^2 + 2 b + b^2 + 2 a (1 + b)) δ (-9 + 2 δ)) ) /. pa → 1/36 (-a + b)
     t (-2 (9 - 5 δ) + 3 (2 + a + b) (9 - 5 δ) √((9 - 2 δ)/(9 - 4 δ)) - 3 (a + b) (3 + δ)) /. {δf → δ, δc → δ}]
```

$$pb \to \frac{1}{9}(a-b)\,t\left(-18+a\,\delta+b\,\delta+2\sqrt{\left(-8+a^2+2b+b^2+2a(1+b)\right)\delta(-9+2\delta)}\right)$$

We check if this derived value of p_B is smaller than p_B-element of point "c3ja2". If the derived value of p_B is actually smaller than p_B-element of point "c3ja2", Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii).
The difference between the derived value of p_B and the p_B-element of c3ja2:

```
Simplify[Factor[1/9 (a - b) t (-18 + a δ + b δ + 2 √((-8 + a^2 + 2 b + b^2 + 2 a (1 + b)) δ (-9 + 2 δ)) ) -
    1/9 (-a + b) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 √((2 - a - b) (4 + a + b) (9 - 2 δ) δ) )]]
```

$$\frac{1}{3}(a-b)\,t\,(6-2\delta+a(-3+2\delta)+b(-3+2\delta))$$

We rearrange this value, and obtain the following:

$$-\frac{1}{3}(b-a)\,t\,(6-2\delta+(a+b)(-3+2\delta))$$

The above is negative. The reason is as follows. 6-2 δ+(a+b) (-3+2 δ) is decreasing in (a+b), hence it is minimized when a+b=2, and then 6-2 δ+2(-3+2 δ)=2δ>0.

Therefore, the difference between the derived value of p_B and the p_B-element of c3ja2 is negative. That is, Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii) if $a + b \geq \frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$.

Thus, for any a, b, and δ, Firm A's "true" reaction function in (iii) does not pass through the reaction function of Firm B in (iii). This shows that the (pure-strategy) pricing equilibrium in period 1 does not exist in (Case iii).

Since (Case i) is symmetric to (Case iii), we can apply the same logic to conclude that the pricing equilibrium in the first period does not exist in (Case i).

We now turn to (Case ii), and identify conditions under which the pricing equilibrium in the first period exists in this case.
We pick up the reaction functions of the firms in (Case ii), without considering if they are in (Case ii). We derive the intersection between the reaction functions:

```
FullSimplify[Solve[{pa ==
    ((9 + 3 δc - 10 δf) pb + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
     (2 (9 + 3 δc - 5 δf)),
   pb == ((9 + 3 δc - 10 δf) pa + (b - a) t (18 + 6 δc - 8 δf + 4 δc δf + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf))) /
     (2 (9 + 3 δc - 5 δf))}, {pa, pb}]]
```

$$\left\{\left\{pa \to \frac{1}{-81+33\,\delta}(a-b)\,t\left(27\,(2+a+b)-6\,(-3+4\,a+4\,b)\,\delta+(-20+9\,a+9\,b)\,\delta^2\right),\right.\right.$$
$$\left.\left.pb \to -\frac{1}{-81+33\,\delta}(a-b)\,t\left(27\,(-4+a+b)-6\,(-5+4\,a+4\,b)\,\delta+(2+9\,a+9\,b)\,\delta^2\right)\right\}\right\}$$

From the result, if there is an intersection between the reaction functions of Firms A and B in (Case ii), the equilibrium p_B is the following:

$$\text{Eq pb}: -\frac{1}{-81+33\,\delta}(a-b)\,t\left(27\,(-4+a+b)-6\,(-5+4\,a+4\,b)\,\delta+(2+9\,a+9\,b)\,\delta^2\right)$$

We can easily show that under the prices the realized z satisfies (a+b)/4<z<(2+a+b)/4 (Case ii).

We rewrite the jump point of Firm A’s reaction function (c3ja1 and c3ja2):

```
c3ja1
c3ja2
```

$$\left\{\frac{1}{36}\,(-a+b)\,t\left(-2\,(9-5\,\delta)+3\,(2+a+b)\,(9-5\,\delta)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-3\,(a+b)\,(3+\delta)\right),\right.$$
$$\left.\frac{1}{18}\,(-a+b)\,t\left(-2-3\,a-3\,b+3\,(2+a+b)\,\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)(9-4\,\delta)\right\}$$

$$\left\{\frac{1}{9}\,(a-b)\,t\left(-18+a\,\delta+b\,\delta+2\,\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}\right),\right.$$
$$\left.\frac{1}{9}\,(-a+b)\,t\left(9\,(4-a-b)-(6-5\,a-5\,b)\,\delta-2\,\sqrt{(2-a-b)\,(4+a+b)\,(9-2\,\delta)\,\delta}\right)\right\}$$

If those values of p_B in the two jump points are larger than the equilibrium p_B in (Case ii), the equilibrium point is stable.

First, we compare p_B at the intersection and at c3ja1:

```
Simplify[Factor[1/18 (-a + b) t (-2 - 3 a - 3 b + 3 (2 + a + b) Sqrt[(9 - 2 δ)/(9 - 4 δ)]) (9 - 4 δ) -
   (-1/(-81 + 33 δ) (a - b) t (27 (-4 + a + b) - 6 (-5 + 4 a + 4 b) δ + (2 + 9 a + 9 b) δ^2))]]
```

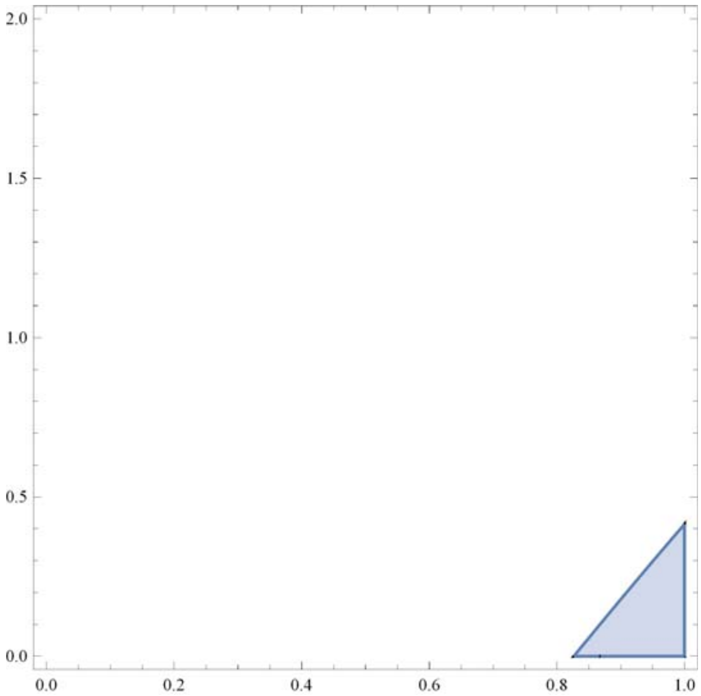
$$\frac{1}{-486+198\,\delta}\,(a-b)\,t\left(3\,a\left(27\left(-7+9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)-3\left(-53+69\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta+\left(-26+44\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta^2\right)+\right.$$
$$3\,b\left(27\left(-7+9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)-3\left(-53+69\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta+\left(-26+44\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta^2\right)+$$
$$\left.2\left(81\left(-7+9\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)+\left(297-621\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta+2\left(-19+66\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)\delta^2\right)\right)$$

Rearranging the above, we obtain

$$\frac{1}{486-198\,\delta}\,(b-a)\,t\left(\left(3\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(9-2\,\delta)\,((63-39\,\delta))\right)(a+b)-\right.$$
$$\left.(9-2\,\delta)\,(126-38\,\delta)+6\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}\right)$$

The sign of this value depends on δ and g≡a+b.
We draw the area in which the above value is negative, that is, the equilibrium point in (Case ii) is not stable.
(Horizontal axis is δ, Vertical axis is g=a+b):

$$\text{RegionPlot}\left[\left(3\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(9-2\,\delta)\,((63-39\,\delta))\right)g-(9-2\,\delta)\,(126-38\,\delta)+6\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}<0\land g<\frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta},\ \{\delta,0,1\},\ \{g,0,2\}\right]$$



On the blue area,the equilibrium point in (Case ii) is **not stable**.

From the figure, we can find the highest value of δ in which the equilibrium point in (Case ii) is stable for any g=a+b, by solving the following equation with respect to δ.

$$\text{NSolve}\left[-(9-2\,\delta)\,(126-38\,\delta)+6\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}==0,\ \delta\right]$$

$$\{\{\delta\to 4.5\},\ \{\delta\to 0.826528\}\}$$

We find that if δ<0.826, the equilibrium point in area (ii) is stable for any g=a+b.

From the figure, we can find the lowest value of g in which the equilibrium point in area (ii) is stable for any δ, by solving the following equation with respect to g.

$$\text{Solve}\left[\left(3\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}-(9-2\,\delta)\,((63-39\,\delta))\right)g-(9-2\,\delta)\,(126-38\,\delta)+6\left(243-207\,\delta+44\,\delta^2\right)\sqrt{\frac{9-2\,\delta}{9-4\,\delta}}==0\ /.\ \delta\to 1,\ g\right]$$

$$\left\{\left\{g\to\frac{77-12\sqrt{35}}{3\left(-7+2\sqrt{35}\right)}\right\}\right\}$$

```
N[%]
```

{{g → 0.414379}}

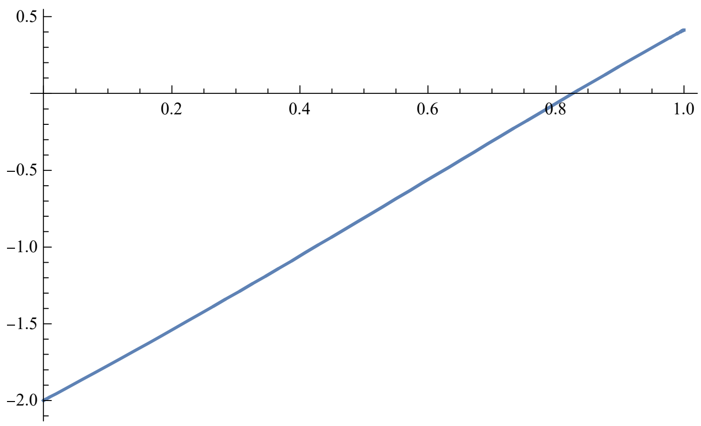
Because of symmetry, the result can be applied to the highest value of g in which the equilibrium point in area (ii) is stable for any δ.

We derive the threshold g as a function of δ.

$$\text{Solve}\left[\left(3\left(243-207\delta+44\delta^2\right)\sqrt{\frac{9-2\delta}{9-4\delta}}-(9-2\delta)\,((63-39\delta))\right)g-(9-2\delta)\,(126-38\delta)+6\left(243-207\delta+44\delta^2\right)\sqrt{\frac{9-2\delta}{9-4\delta}}==0,\ g\right]$$

$$\left\{\left\{g\to\frac{(126-38\delta)\,(9-2\delta)-6\sqrt{\frac{9-2\delta}{9-4\delta}}\left(243-207\delta+44\delta^2\right)}{-(63-39\delta)\,(9-2\delta)+3\sqrt{\frac{9-2\delta}{9-4\delta}}\left(243-207\delta+44\delta^2\right)}\right\}\right\}$$

$$\text{Plot}\left[\left(-12\,(3-\delta)\left(1296-1431\delta+479\delta^2\right)+120\,\delta\,(27-11\delta)\sqrt{(9-4\delta)\,(9-2\delta)}\right)\Big/\left(18\left(1296-1233\delta+426\delta^2-73\delta^3\right)\right),\ \{\delta,0,1\}\right]$$



$$N\left[2-\frac{77-12\sqrt{35}}{3\left(-7+2\sqrt{35}\right)}\right]$$

1.58562

Thus, we find that if 0.415<g<1.585, the equilibrium point in area (ii) is stable for any δ.

$$\left\{\frac{1}{9}\,(a-b)\,t\left(-18+a\,\delta+b\,\delta+2\sqrt{\left(-8+a^2+2b+b^2+2a\,(1+b)\right)\delta\,(-9+2\delta)}\right),\right.$$
$$\left.\frac{1}{9}\,(-a+b)\,t\left(9\,(4-a-b)-(6-5a-5b)\,\delta-2\sqrt{(2-a-b)\,(4+a+b)\,(9-2\delta)\,\delta}\right)\right\}$$

Second, we compare p_B at the intersection and at c3ja2:

```
Simplify[Factor[1/9 (-a + b) t (9 (4 - a - b) - (6 - 5 a - 5 b) δ - 2 √((2 - a - b) (4 + a + b) (9 - 2 δ) δ)) -
    (-1/(-81 + 33 δ) (a - b) t (27 (-4 + a + b) - 6 (-5 + 4 a + 4 b) δ + (2 + 9 a + 9 b) δ^2))]]
```

$$-\frac{1}{-243+99\,\delta}\,2\,(a-b)\,t\left(-324+81\,b+234\,\delta-81\,b\,\delta-36\,\delta^2+14\,b\,\delta^2+27\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}-11\,\delta\sqrt{\left(-8+a^2+2\,b+b^2+2\,a\,(1+b)\right)\delta\,(-9+2\,\delta)}+a\left(81-81\,\delta+14\,\delta^2\right)\right)$$

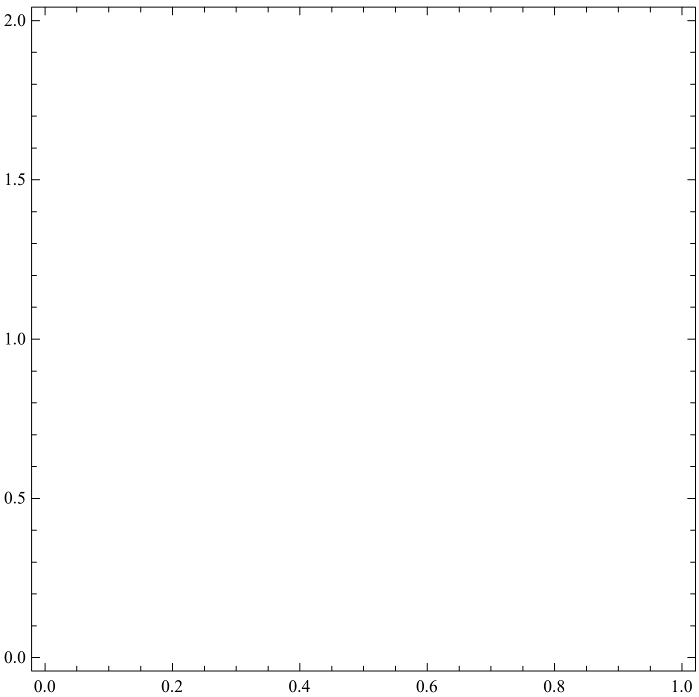
Rearranging the above, we obtain

$$-\frac{1}{243-99\,\delta}\,2\,(b-a)\,t\left(-18\,(2-\delta)\,(9-2\,\delta)+(27-11\,\delta)\sqrt{(2-(a+b))\,(4+a+b)\,\delta\,(9-2\,\delta)}+(a+b)\,(9-2\,\delta)\,(9-7\,\delta)\right)$$

The sign of this value depends on δ and g=a+b.
We draw the area in which the above value is negative, that is, the equilibrium point in (Case ii) is not stable.
(Horizontal axis is δ, Vertical axis is g=a+b):

$$g>\frac{2\left(-18-\delta+6\sqrt{81-54\,\delta+8\,\delta^2}\right)}{36-7\,\delta}$$

```
RegionPlot[(-18 (2 - δ) (9 - 2 δ) + (27 - 11 δ) √((2 - g) (4 + g) δ (9 - 2 δ)) + g (9 - 2 δ) (9 - 7 δ)) > 0,
  {δ, 0, 1}, {g, 0, 2}]
```



This means that Firm B's deviation incentive does not matter.

From the two arguments, we find that if 0.415<g<1.585, the equilibrium point in area (ii) is stable for any δ.

Thus we conclude that (i) if $\delta < 0.826$, the pricing equilibrium exists in (Case ii) for any a + b, and (ii) if $0.415 < a+b < 1.585$, then the pricing equilibrium exists in (Case ii) for any δ.

3. First Period - Locations

The profits of Firms A and B are respectively

```
Simplify[Factor[pa z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
   z → 3 (pb - pa)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → ((a - b) t
      (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf)) /
    (81 + 27 δc - 60 δf), pb → -(((a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
         4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)) / (81 + 27 δc - 60 δf))}]]
Simplify[Factor[pb (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
   z → 3 (pb - pa)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → ((a - b) t
      (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf)) /
    (81 + 27 δc - 60 δf), pb → -(((a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
         4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)) / (81 + 27 δc - 60 δf))}]]
```

$$-\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}(a-b)\,t\,\Big(9\,a^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+9\,b^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)-12\,b\left(-243+18\,\delta c^3+\delta c^2\,(27-57\,\delta f)+216\,\delta f+84\,\delta f^2-80\,\delta f^3+\delta c\left(-162-9\,\delta f+88\,\delta f^2\right)\right)+4\left(729+108\,\delta c^3-972\,\delta f+216\,\delta f^2+80\,\delta f^3-9\,\delta c^2\,(-72+43\,\delta f)+3\,\delta c\left(405-450\,\delta f+104\,\delta f^2\right)\right)+6\,a\Big(3\,b\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)-2\left(-243+18\,\delta c^3+\delta c^2\,(27-57\,\delta f)+216\,\delta f+84\,\delta f^2-80\,\delta f^3+\delta c\left(-162-9\,\delta f+88\,\delta f^2\right)\right)\Big)\Big)$$

$$-\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}(a-b)\,t\,\Big(9\,a^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+9\,b^2\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+4\left(2916+27\,\delta c^3-1863\,\delta f-1944\,\delta f^2+1280\,\delta f^3+9\,\delta c^2\,(45+16\,\delta f)-18\,\delta c\left(-108+3\,\delta f+56\,\delta f^2\right)\right)+12\,b\left(-486+9\,\delta c^3+81\,\delta f+636\,\delta f^2-320\,\delta f^3-6\,\delta c^2\,(-9+20\,\delta f)+\delta c\left(-81-423\,\delta f+352\,\delta f^2\right)\right)+6\,a\Big(3\,b\left(81+3\,\delta c^3+45\,\delta f-184\,\delta f^2+80\,\delta f^3+3\,\delta c^2\,(-3+7\,\delta f)+\delta c\left(-27+138\,\delta f-88\,\delta f^2\right)\right)+2\left(-486+9\,\delta c^3+81\,\delta f+636\,\delta f^2-320\,\delta f^3-6\,\delta c^2\,(-9+20\,\delta f)+\delta c\left(-81-423\,\delta f+352\,\delta f^2\right)\right)\Big)\Big)$$

Rearranging them, we obtain the profits of Firms A and B respectively

$$\frac{(b-a)\,t}{18\,(27+9\,\delta c-20\,\delta f)^2}\Big(9\,(a+b)^2\left(3\,\delta c^3-3\,\delta c^2\,(3-7\,\delta f)-\delta c\left(27-138\,\delta f+88\,\delta f^2\right)+81+45\,\delta f-184\,\delta f^2+80\,\delta f^3\right)+12\,(a+b)\left(-18\,\delta c^3-3\,\delta c^2\,(9-19\,\delta f)+\delta c\left(162+9\,\delta f-88\,\delta f^2\right)+243-216\,\delta f-84\,\delta f^2+80\,\delta f^3\right)+4\left(108\,\delta c^3+9\,\delta c^2\,(72-43\,\delta f)+3\,\delta c\left(405-450\,\delta f+104\,\delta f^2\right)+729-972\,\delta f+216\,\delta f^2+80\,\delta f^3\right)\Big)$$

$$\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}(b-a)\,t\,\Big(9\,(a+b)^2\left(3\,\delta c^3-3\,\delta c^2\,(3-7\,\delta f)-\delta c\left(27-138\,\delta f+88\,\delta f^2\right)+81+45\,\delta f-184\,\delta f^2+80\,\delta f^3\right)+12\,(a+b)\left(9\,\delta c^3+6\,\delta c^2\,(9-20\,\delta f)-\delta c\left(81+423\,\delta f-352\,\delta f^2\right)-486+81\,\delta f+636\,\delta f^2-320\,\delta f^3\right)+4\left(27\,\delta c^3+9\,\delta c^2\,(45+16\,\delta f)+18\,\delta c\left(108-3\,\delta f-56\,\delta f^2\right)+2916-1863\,\delta f-1944\,\delta f^2+1280\,\delta f^3\right)\Big)$$

We define J1, the coefficient of $(a+b)^2$, as follows:

```
J1 = (3 δc^3 - 3 δc^2 (3 - 7 δf) - δc (27 - 138 δf + 88 δf^2) + 81 + 45 δf - 184 δf^2 + 80 δf^3)
```

$$81+3\,\delta c^3-3\,\delta c^2\,(3-7\,\delta f)+45\,\delta f-184\,\delta f^2+80\,\delta f^3-\delta c\,(27-138\,\delta f+88\,\delta f^2)$$

Differentiating Firm A's profit with respect to a and Firm B's profit with respect to b, we obtain

```
Factor[D[pa z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
    z → 3 (pb - pa)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /.
   {pa → ((a - b) t (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 -
          4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf)) / (81 + 27 δc - 60 δf),
    pb → -(((a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 - 4 (21 + 3 a (-2 + δc) + 3
              b (-2 + δc) + 2 δc) δf)) / (81 + 27 δc - 60 δf))}, a]]
Factor[D[pb (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
    z → 3 (pb - pa)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → ((a - b) t
       (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf)) /
     (81 + 27 δc - 60 δf), pb → -(((a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
          4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)) / (81 + 27 δc - 60 δf))}, b]]
```

$$-\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}\,t\,\big(2916+5832\,a+2187\,a^2+1458\,a\,b-729\,b^2+4860\,\delta c+3888\,a\,\delta c-729\,a^2\,\delta c-486\,a\,b\,\delta c+243\,b^2\,\delta c+2592\,\delta c^2-648\,a\,\delta c^2-243\,a^2\,\delta c^2-162\,a\,b\,\delta c^2+81\,b^2\,\delta c^2+432\,\delta c^3-432\,a\,\delta c^3+81\,a^2\,\delta c^3+54\,a\,b\,\delta c^3-27\,b^2\,\delta c^3-3888\,\delta f-5184\,a\,\delta f+1215\,a^2\,\delta f+810\,a\,b\,\delta f-405\,b^2\,\delta f-5400\,\delta c\,\delta f+216\,a\,\delta c\,\delta f+3726\,a^2\,\delta c\,\delta f+2484\,a\,b\,\delta c\,\delta f-1242\,b^2\,\delta c\,\delta f-1548\,\delta c^2\,\delta f+1368\,a\,\delta c^2\,\delta f+567\,a^2\,\delta c^2\,\delta f+378\,a\,b\,\delta c^2\,\delta f-189\,b^2\,\delta c^2\,\delta f+864\,\delta f^2-2016\,a\,\delta f^2-4968\,a^2\,\delta f^2-3312\,a\,b\,\delta f^2+1656\,b^2\,\delta f^2+1248\,\delta c\,\delta f^2-2112\,a\,\delta c\,\delta f^2-2376\,a^2\,\delta c\,\delta f^2-1584\,a\,b\,\delta c\,\delta f^2+792\,b^2\,\delta c\,\delta f^2+320\,\delta f^3+1920\,a\,\delta f^3+2160\,a^2\,\delta f^3+1440\,a\,b\,\delta f^3-720\,b^2\,\delta f^3\big)$$

$$-\frac{1}{18\,(27+9\,\delta c-20\,\delta f)^2}\,t\,\big(-11\,664+729\,a^2+11\,664\,b-1458\,a\,b-2187\,b^2-7776\,\delta c-243\,a^2\,\delta c+1944\,b\,\delta c+486\,a\,b\,\delta c+729\,b^2\,\delta c-1620\,\delta c^2-81\,a^2\,\delta c^2-1296\,b\,\delta c^2+162\,a\,b\,\delta c^2+243\,b^2\,\delta c^2-108\,\delta c^3+27\,a^2\,\delta c^3-216\,b\,\delta c^3-54\,a\,b\,\delta c^3-81\,b^2\,\delta c^3+7452\,\delta f+405\,a^2\,\delta f-1944\,b\,\delta f-810\,a\,b\,\delta f-1215\,b^2\,\delta f+216\,\delta c\,\delta f+1242\,a^2\,\delta c\,\delta f+10\,152\,b\,\delta c\,\delta f-2484\,a\,b\,\delta c\,\delta f-3726\,b^2\,\delta c\,\delta f-576\,\delta c^2\,\delta f+189\,a^2\,\delta c^2\,\delta f+2880\,b\,\delta c^2\,\delta f-378\,a\,b\,\delta c^2\,\delta f-567\,b^2\,\delta c^2\,\delta f+7776\,\delta f^2-1656\,a^2\,\delta f^2-15\,264\,b\,\delta f^2+3312\,a\,b\,\delta f^2+4968\,b^2\,\delta f^2+4032\,\delta c\,\delta f^2-792\,a^2\,\delta c\,\delta f^2-8448\,b\,\delta c\,\delta f^2+1584\,a\,b\,\delta c\,\delta f^2+2376\,b^2\,\delta c\,\delta f^2-5120\,\delta f^3+720\,a^2\,\delta f^3+7680\,b\,\delta f^3-1440\,a\,b\,\delta f^3-2160\,b^2\,\delta f^3\big)$$

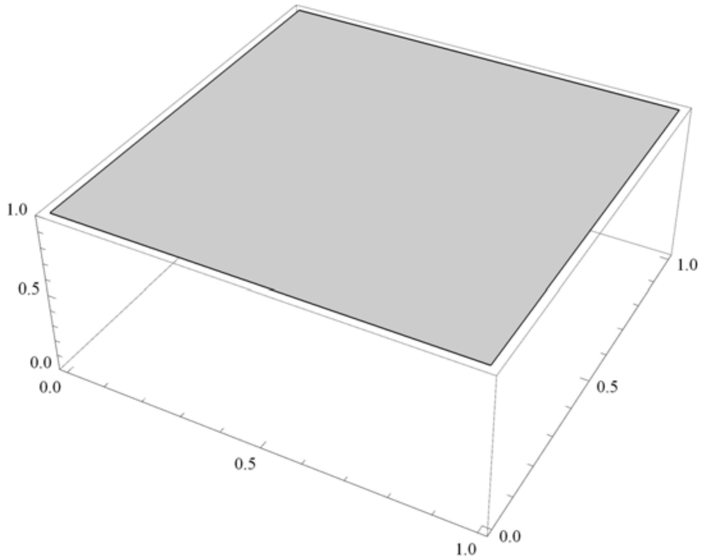
The above first-order derivatives can be rearranged as follows.

$$-\frac{t}{18\,(27+9\,\delta c-20\,\delta f)^2}\Big(27\,J1\,a^2+\big(24\,(9\,(9-\delta c^2)\,(3+2\,\delta c)-3\,(72-3\,\delta c-19\,\delta c^2)\,\delta f-4\,(21+22\,\delta c)\,\delta f^2+80\,\delta f^3)+18\,J1\,b\big)\,a+4\,(27\,(3+\delta c)\,(3+2\,\delta c)^2-9\,(108+150\,\delta c+43\,\delta c^2)\,\delta f+24\,(9+13\,\delta c)\,\delta f^2+80\,\delta f^3)-9\,J1\,b^2\Big)$$

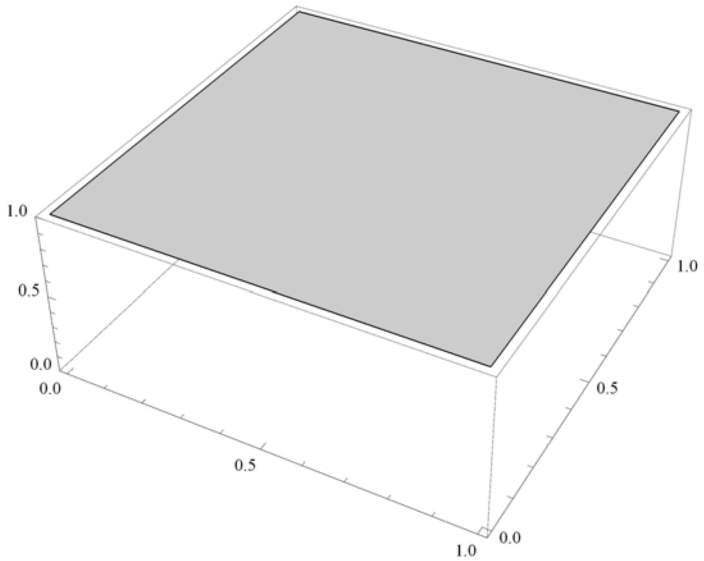
$$\frac{t}{18\,(27+9\,\delta c-20\,\delta f)^2}\Big(27\,J1\,b^2+\big(24\,(-9\,(9-\delta c^2)\,(6+\delta c)+3\,(27-\delta c\,(141+40\,\delta c))\,\delta f+4\,(159+88\,\delta c)\,\delta f^2-320\,\delta f^3)+18\,J1\,a\big)\,b+4\,(27\,(3+\delta c)\,(6+\delta c)^2-9\,(207+2\,\delta c\,(3-8\,\delta c))\,\delta f-72\,(27+14\,\delta c)\,\delta f^2+1280\,\delta f^3)-9\,J1\,a^2\Big)$$

First, we show that the derivative of Firm A's profit monotonically decreases in a. This implies that Firm A's optimal location choice is $a = 0$. To show this, we show that the coefficients to a^2, *a* and the constant term in Firm A's first-order derivative are all positive. For the coefficients to a^2 and *a*, see the following figures, which show they are both positive.

```
Plot3D[J1, {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```

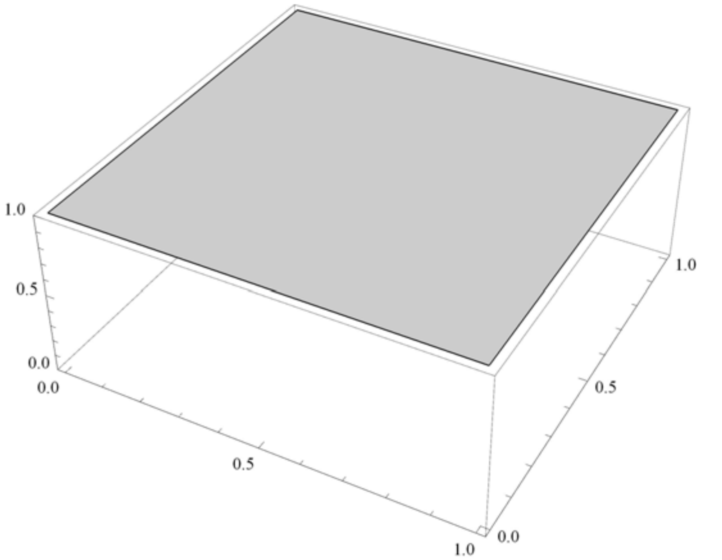



```
Plot3D[24 (9 (9 - δc^2) (3 + 2 δc) - 3 (72 - 3 δc - 19 δc^2) δf - 4 (21 + 22 δc) δf^2 + 80 δf^3),
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



Those are positive.
For the constant term in Firm A's first-order derivative, notice that it is minimized when $b = 1$.
As the next figure shows, the constant term is positive when $b = 1$.

```
Plot3D[4 (27 (3 + δc) (3 + 2 δc)^2 - 9 (108 + 150 δc + 43 δc^2) δf + 24 (9 + 13 δc) δf^2 + 80 δf^3) - 9 J1,
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



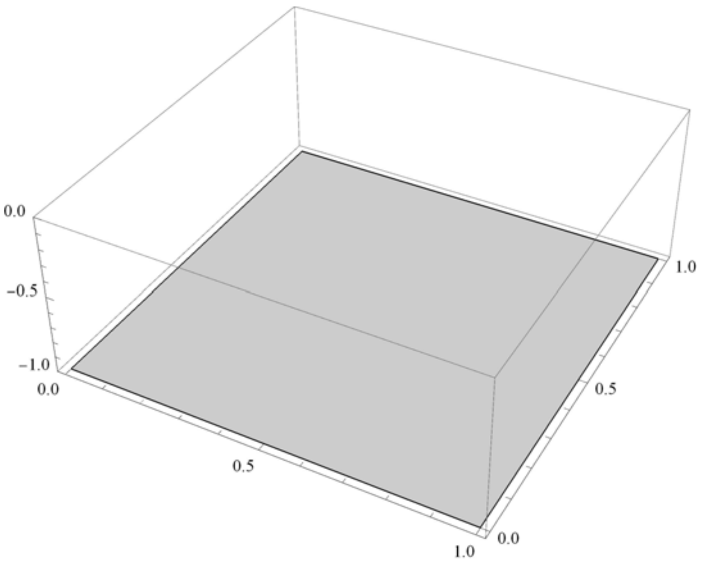
Put together, we have shown that Firm A's profit decreases monotonically in a. Thus $a = 0$ is Firm A's optimal location choice.

Next, we turn to Firm B's problem. Differentiating the numerator of the first-order derivative of Firm B's profit with respect to b, we obtain

```
54 J1 b +
 (24 (-9 (9 - δc^2) (6 + δc) + 3 (27 - δc (141 + 40 δc)) δf + 4 (159 + 88 δc) δf^2 - 320 δf^3) + 18 J1 a)
```

This is maximized when a=b=1. At a=b=1, we can show that this is negative (see the following figure).

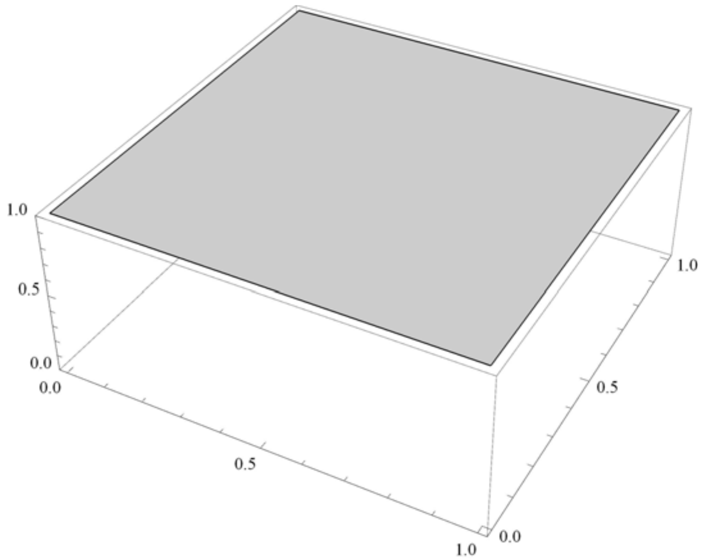
```
Plot3D[
 54 J1 + (24 (-9 (9 - δc^2) (6 + δc) + 3 (27 - δc (141 + 40 δc)) δf + 4 (159 + 88 δc) δf^2 - 320 δf^3) + 18 J1),
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {-1, 0}]
```



Thus the first-order derivative of Firm B's profit decreases monotonically in b.

We now show that at a=0 and b=1, the numerator of the first-order derivative of Firm B's profit is positive (see the following figure). This implies that Firm B's optimal location choice is $b = 1$.

```
Plot3D[
 (27 J1 + (24 (-9 (9 - δc²) (6 + δc) + 3 (27 - δc (141 + 40 δc)) δf + 4 (159 + 88 δc) δf² - 320 δf³) ) +
   4 (27 (3 + δc) (6 + δc)² - 9 (207 + 2 δc (3 - 8 δc)) δf - 72 (27 + 14 δc) δf² + 1280 δf³)),
 {δc, 0, 1}, {δf, 0, 1}, PlotRange → {0, 1}]
```



Therefore, the equilibrium locations are a=0, b=1.
The rest of Proposition 1 follows by substituting $a = 0$, $b = 1$ into relevant prices and the locations of marginal consumers in the two periods.

Proofs of Propositions 2 and 3.

Because each firm's reaction function consists of three different pieces, we need to derive the 'true' reaction function by checking when the firm's profit obtains a global, rather than, local maximum.

From here on, we assume that firms' discount factor is δ and consumers' discount factor is 0, as stated in Propositions 2 and 3.

```
δf = δ
```

δ

```
δc = 0
```

0

From Firm A's reaction function derived in the proof of Proposition 1, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm A's reaction function, and the second endpoint below locates the right-hand side of Firm A's reaction function in the first period.

$$\text{c1aL} = \text{Simplify}\left[\left\{\frac{1}{9}\,(a-b)\,(-8+5\,a+5\,b)\,t\,\delta f,\ -\frac{1}{9}\,(b-a)\,t\,((a+b)\,(9-6\,\delta c+5\,\delta f)+6\,\delta c-8\,\delta f)\right\}\right]$$

$$\text{c1aR} = \text{FullSimplify}\left[\text{Factor}\left[\left\{\frac{1}{18}\,(a-b)\,t\,(3\,(a+b)\,(-3+\delta c)-16\,\delta f),\ \frac{(b-a)\,t\,(3\,(-2+a+b)\,\delta c+8\,\delta f)}{9}\right\}\right]\right]$$

$$\left\{\frac{1}{9}\,(a-b)\,(-8+5\,a+5\,b)\,t\,\delta,\ -\frac{1}{9}\,(-a+b)\,t\,(-8\,\delta+(a+b)\,(9+5\,\delta))\right\}$$

$$\left\{-\frac{1}{18}\,(a-b)\,t\,(9\,(a+b)+16\,\delta),\ \frac{8}{9}\,(-a+b)\,t\,\delta\right\}$$

(Case ii): The first endpoint below locates the left-hand side of Firm A’s reaction function, and the second endpoint below locates the right-hand side of Firm A’s reaction function in the first period.

$$\text{c2aL} = \text{FullSimplify}\left[\text{Factor}\left[\left\{-\frac{1}{18}\,(a-b)\,t\,(3\,(a+b)\,(3+\delta c)-2\,(-8+3\,a+3\,b)\,\delta f),\ \frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)-2\,(3\,\delta c-4\,\delta f))}{9}\right\}\right]\right]$$

$$\text{c2aR} = \text{Simplify}\left[\left\{-\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(3+\delta c)-2\,(2+3\,a+3\,b)\,\delta f),\ \frac{(b-a)\,t\,(3\,(a+b)\,(2\,\delta c-\delta f)+2\,(9-\delta f))}{9}\right\}\right]$$

$$\left\{\frac{1}{18}\,(a-b)\,t\,(-9\,(a+b)+2\,(-8+3\,a+3\,b)\,\delta),\ \frac{1}{9}\,(a-b)\,(-8+3\,a+3\,b)\,t\,\delta\right\}$$

$$\left\{-\frac{1}{18}\,(a-b)\,t\,(9\,(2+a+b)-2\,(2+3\,a+3\,b)\,\delta),\ \frac{1}{9}\,(a-b)\,t\,(-18+(2+3\,a+3\,b)\,\delta)\right\}$$

At this stage, we cannot say if Firm A’s reaction function is also continuous in (Case i) and (Case ii). We will come back to this shortly. Note that the left-hand endpoint in (Case i) corresponds to (a+b)/4=z.

(Case iii): The first endpoint below locates the left-hand side of Firm A’s reaction function, and the second endpoint below locates the right-hand side of Firm A’s reaction function in in the first period.

$$\text{c3aL} = \text{Simplify}\left[\left\{\frac{1}{18}\,(a-b)\,t\,(3\,(2+a+b)\,(-3+\delta c)+2\,(2+3\,a+3\,b)\,\delta f),\ \frac{(b-a)\,t\,(2\,(9-3\,\delta c-\delta f)+3\,(a+b)\,(\delta c-\delta f))}{9}\right\}\right]$$

$$\text{c3aR} = \text{Simplify}\left[\left\{\frac{2}{9}\,(a-b)\,t\,(-9+3\,\delta c+(2+a+b)\,\delta f),\ \frac{(b-a)\,t\,(4\,(9-3\,\delta c-\delta f)-(a+b)\,(9-6\,\delta c+2\,\delta f))}{9}\right\}\right]$$

$$\left\{\frac{1}{18}\,(a-b)\,t\,(-9\,(2+a+b)+2\,(2+3\,a+3\,b)\,\delta),\ \frac{1}{9}\,(a-b)\,t\,(-18+(2+3\,a+3\,b)\,\delta)\right\}$$

$$\left\{\frac{2}{9}\,(a-b)\,t\,(-9+(2+a+b)\,\delta),\ \frac{1}{9}\,(-a+b)\,t\,(-4\,(-9+\delta)-(a+b)\,(9+2\,\delta))\right\}$$

As shown above, the left endpoint of Firm A’s reaction function in (Case iii) coincides with the right endpoint of Firm A’s reaction function in (Case ii). That is, Firm A’s reaction function is continuous in (Case ii) and (Case iii).

(Case iii)’: When z=1, the reaction function of Firm A consists of the segment connecting the following two points.

```
c3daL = Simplify[
  {2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf), ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9}]
c3daR = Simplify[{2/9 (a - b) t (-9 + 3 δc + (2 + a + b) δf) + k,
   ((b - a) t (4 (9 - 3 δc - δf) - (a + b) (9 - 6 δc + 2 δf)))/9 + k}]
```

$\left\{\frac{2}{9}(a-b)\,t\,(-9+(2+a+b)\,\delta),\ \frac{1}{9}(-a+b)\,t\,(-4\,(-9+\delta)-(a+b)\,(9+2\,\delta))\right\}$

$\left\{k+\frac{2}{9}(a-b)\,t\,(-9+(2+a+b)\,\delta),\ k+\frac{1}{9}(-a+b)\,t\,(-4\,(-9+\delta)-(a+b)\,(9+2\,\delta))\right\}$

where k is a sufficient large positive number (to keep p_A at the monopoly price leading to $z=1$). Note that the left endpoint of Firm A's reaction function in (Case iii)' coincides with the right endpoint of Firm A's reaction function in (Case iii).

Next, from firm B's reaction function derived the proof of Proposition 1, we check the endpoints of each of the three line segments corresponding to the three cases: (i)0≤z≤(a+b)/4, (ii) (a+b)/4<z<(2+a+b)/4, (iii) (2+a+b)/4≤z≤1.

(Case i): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in the first period.

```
c1bL = FullSimplify[{((b - a) t (18 - 3 (a + b) (δc - δf) - 8 δf))/9,
   -1/18 (a - b) t (3 (-4 + a + b) (-3 + δc) + 2 (-8 + 3 a + 3 b) δf)}]
c1bR = Simplify[{((b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf))/9,
   -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf)}]
```

$\left\{\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+3\,(a+b)\,\delta),\ -\frac{1}{18}(a-b)\,t\,(-9\,(-4+a+b)+2\,(-8+3\,a+3\,b)\,\delta)\right\}$

$\left\{\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+(a+b)\,(9+2\,\delta)),\ -\frac{2}{9}(a-b)\,t\,(9+(-4+a+b)\,\delta)\right\}$

(Case i)': When z=0, The reaction function of Firm B consists of the segment connecting the following two points.

```
c1dbL =
 Simplify[{((b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf))/9, -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf)}]
c1dbR = Simplify[{((b - a) t (18 + (a + b) (9 - 6 δc + 2 δf) - 8 δf))/9 + k,
   -2/9 (a - b) t (9 - 3 δc + (-4 + a + b) δf) + k}]
```

$\left\{\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+(a+b)\,(9+2\,\delta)),\ -\frac{2}{9}(a-b)\,t\,(9+(-4+a+b)\,\delta)\right\}$

$\left\{k+\frac{1}{9}(-a+b)\,t\,(18-8\,\delta+(a+b)\,(9+2\,\delta)),\ k-\frac{2}{9}(a-b)\,t\,(9+(-4+a+b)\,\delta)\right\}$

where k is a sufficient large positive number (to keep p_B at the monopoly price leading to $z=0$). Note that the left-hand endpoint of Firm A's reaction function in case (i)' coincides with the right-hand endpoint of Firm A's reaction function in case (i).

Case (ii): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in the first period.

```
c2bL = FullSimplify[{(b - a) t (6 (1 - a - b) δc + (2 + 3 a + 3 b) δf)/9,
   1/18 (a - b) t (3 (-2 + a + b) (3 + δc) - 2 (2 + 3 a + 3 b) δf)}]
c2bR = Simplify[{(b - a) t (18 + 6 (2 - a - b) δc - (8 - 3 a - 3 b) δf)/9,
   1/18 (a - b) t (3 (-4 + a + b) (3 + δc) - 2 (-8 + 3 a + 3 b) δf)}]
```

$\left\{\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-2(2+3a+3b)\,\delta)\right\}$

$\left\{-\frac{1}{9}(a-b)\,t\,(18+(-8+3a+3b)\,\delta),\ \frac{1}{18}(a-b)\,t\,(9(-4+a+b)-2(-8+3a+3b)\,\delta)\right\}$

We find that the left endpoint of Firm B's reaction function in (Case i) coincides with the right endpoint of Firm B's reaction function in (Case ii). That is, Firm B's reaction function is continuous in (Case i) and (Case ii).

Case (iii): The first endpoint below locates the left-hand side of Firm B's reaction function, and the second endpoint below locates the right-hand side of Firm B's reaction function in the first period.

```
c3bL =
  Simplify[{(b - a) t ((a + b) (9 - 6 δc + 5 δf) - 2 (9 - 3 δc + δf))/9, -1/9 (a - b) (-2 + 5 a + 5 b) t δf}]
c3bR = FullSimplify[
  Factor[{(b - a) t (8 δf - 3 (a + b) δc)/9, -1/18 (a - b) t (3 (-2 + a + b) (-3 + δc) + 16 δf)}]]
```

$\left\{\frac{1}{9}(-a+b)\,t\,(-2(9+\delta)+(a+b)(9+5\delta)),\ -\frac{1}{9}(a-b)(-2+5a+5b)\,t\,\delta\right\}$

$\left\{\frac{8}{9}(-a+b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-16\,\delta)\right\}$

At this stage, we cannot say if Firm B's reaction function is continuous in (Case iii) and (Case ii). We will come back to this shortly. Note that the right-hand endpoint in (Case iii) corresponds to (2+a+b)/4=z.

We now turn to the pricing equilibrium in the first period. To this end, we need to find the 'true' reaction function for each firm, by checking when each firm's reaction leads to a global optimum. Next we check when the two 'true' reaction functions intersect. As before, calculations are quite messy although the logical steps are identical. If necessary, readers can jump directly to the stage where we show that the pricing equilibrium exists only in (Case ii) and for all values of δ.

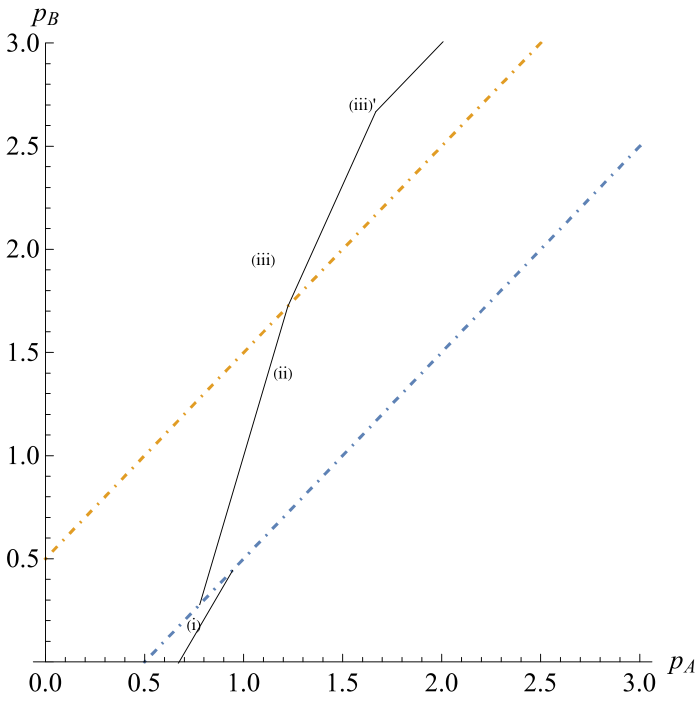
For illustrative purposes, we start with an example where we set a = 0, b = 1, t = 1, and δ = 1/2.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 1
```

```
0

1

1
─
2

1

1
```

First, we plot Firm A’s reaction function corresponding to the three cases.

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1aR}], Line[{c2aL, c2aR}], , Line[{c3aL, c3aR}],
    Line[{c3daL, c3daR}], Text["(iii)'", {1.6, 2.7}], Text["(iii)", {1.1, 1.95}],
    Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



As shown above, for some p_B, there are two local optimal prices for Firm A.
We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the three endpoints: The left-hand endpoint in (ii) (c2aL), the left-hand and right-hand endpoints (c1aL and c1aR) in (i) (see below)

```
c2aL
c1aL
c1aR
```

$\left\{\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\delta),\ \frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta\right\}$

$\left\{\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta,\ -\frac{1}{9}(-a+b)\,t\,(-8\delta+(a+b)(9+5\delta))\right\}$

$\left\{-\frac{1}{18}(a-b)\,t\,(9(a+b)+16\delta),\ \frac{8}{9}(-a+b)\,t\,\delta\right\}$

First, we compare the elements of the left-hand endpoint in (ii) and the right-hand endpoint in (i):

$\text{Factor}\left[\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\delta)-\left(-\frac{1}{18}(a-b)\,t\,(9(a+b)+16\delta)\right)\right]$

$\text{Factor}\left[\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta-\frac{8}{9}(-a+b)\,t\,\delta\right]$

$\frac{1}{3}(a-b)(a+b)\,t\,\delta$

$\frac{1}{3}(a-b)(a+b)\,t\,\delta$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the left-hand endpoint in (ii) is located below the right-hand endpoint in (i) as in the above Figure.

Second, we compare the elements of the left-hand endpoint in (ii) and the left-hand endpoint in (i):

$\text{Factor}\left[\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\delta)-\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta\right]$

$\text{Factor}\left[\frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta-\left(-\frac{1}{9}(-a+b)\,t\,(-8\delta+(a+b)(9+5\delta))\right)\right]$

$-\frac{1}{18}(a-b)(a+b)\,t\,(9+4\delta)$

$-\frac{1}{9}(a-b)(a+b)\,t\,(9+2\delta)$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the left-hand endpoint in (ii) is located above the left-hand endpoint in (i) as in the above Figure.

We need to find the global optimal price of Firm A, p_A, when there are two local optima for a given p_B. There is a price p_B such that choosing the reaction function in (ii) and choosing the reaction function in (i) are indifferent for Firm A. This p_B is the threshold for which choosing the reaction function in (ii) is preferred by Firm A if p_B is larger than this threshold p_B, otherwise, choosing the reaction function in (i) is preferred by Firm A. We need to find the threshold value of p_B.

To check the threshold value of p_B for Firm A's reaction function, we derive the profits under cases (ii) and (i).
The interior profit of firm A under case (ii) for p_B is

```
Factor[pA z + δf 1/9 (b - a) t (2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
   z → 3 (pB - pA)/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pA →
    ((9 + 3 δc - 10 δf) pB + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf)))/
     (2 (9 + 3 δc - 5 δf))}]
```

$$-\frac{1}{72\,(a-b)\,t\,(-9+5\,\delta)}\left(-81\,a^4\,t^2+162\,a^2\,b^2\,t^2-81\,b^4\,t^2-144\,a^2\,t^2\,\delta-36\,a^4\,t^2\,\delta+288\,a\,b\,t^2\,\delta-144\,b^2\,t^2\,\delta+72\,a^2\,b^2\,t^2\,\delta-36\,b^4\,t^2\,\delta+16\,a^2\,t^2\,\delta^2+48\,a^3\,t^2\,\delta^2+36\,a^4\,t^2\,\delta^2-32\,a\,b\,t^2\,\delta^2-48\,a^2\,b\,t^2\,\delta^2+16\,b^2\,t^2\,\delta^2-48\,a\,b^2\,t^2\,\delta^2-72\,a^2\,b^2\,t^2\,\delta^2+48\,b^3\,t^2\,\delta^2+36\,b^4\,t^2\,\delta^2+162\,a^2\,t\,p_B-162\,b^2\,t\,p_B-144\,a\,t\,\delta\,p_B-36\,a^2\,t\,\delta\,p_B+144\,b\,t\,\delta\,p_B+36\,b^2\,t\,\delta\,p_B-81\,p_B^2\right)$$

The interior profit of firm A under case (i) for p_B is

```
Factor[pA z + δf (1/18 (b - a) t (4 + 4 a + a^2 + 4 b + 2 a b + b^2 - 16 z + 10 a z + 10 b z - 20 z^2)) /.
   z → (-3 a^2 t + 3 b^2 t - 2 a t δc + 2 a^2 t δc + 2 b t δc - 2 b^2 t δc - 3 pA + 3 pB) / (2 (a - b) t (-3 + δc)) /.
  pA → 1/(2 (-9 + 3 δc - 5 δf)) ((a - b) t (6 δc - 2 δc^2 + 8 δf + 4 δc δf + b (9 - 9 δc + 2 δc^2 + 5 δf - 5 δc δf) +
        a (9 + 2 δc^2 + 5 δf - δc (9 + 5 δf))) + (-9 + 3 δc - 10 δf) pB)]
```

$$-\frac{1}{8\,(a-b)\,t\,(9+5\,\delta)}\left(9\,a^4\,t^2-18\,a^2\,b^2\,t^2+9\,b^4\,t^2+16\,a^2\,t^2\,\delta+14\,a^4\,t^2\,\delta-32\,a\,b\,t^2\,\delta+16\,b^2\,t^2\,\delta-28\,a^2\,b^2\,t^2\,\delta+14\,b^4\,t^2\,\delta+16\,a^2\,t^2\,\delta^2+5\,a^4\,t^2\,\delta^2-32\,a\,b\,t^2\,\delta^2+16\,b^2\,t^2\,\delta^2-10\,a^2\,b^2\,t^2\,\delta^2+5\,b^4\,t^2\,\delta^2-18\,a^2\,t\,p_B+18\,b^2\,t\,p_B+16\,a\,t\,\delta\,p_B-10\,a^2\,t\,\delta\,p_B-16\,b\,t\,\delta\,p_B+10\,b^2\,t\,\delta\,p_B+9\,p_B^2\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

```
FullSimplify[
 Solve[{-1/(72 (a - b) t (-9 + 5 δ)) (-81 a^4 t^2 + 162 a^2 b^2 t^2 - 81 b^4 t^2 - 144 a^2 t^2 δ - 36 a^4 t^2 δ + 288 a b t^2 δ -
       144 b^2 t^2 δ + 72 a^2 b^2 t^2 δ - 36 b^4 t^2 δ + 16 a^2 t^2 δ^2 + 48 a^3 t^2 δ^2 + 36 a^4 t^2 δ^2 - 32 a b t^2 δ^2 -
       48 a^2 b t^2 δ^2 + 16 b^2 t^2 δ^2 - 48 a b^2 t^2 δ^2 - 72 a^2 b^2 t^2 δ^2 + 48 b^3 t^2 δ^2 + 36 b^4 t^2 δ^2 +
       162 a^2 t pB - 162 b^2 t pB - 144 a t δ pB - 36 a^2 t δ pB + 144 b t δ pB + 36 b^2 t δ pB - 81 pB^2) ==
    -1/(8 (a - b) t (9 + 5 δ)) (9 a^4 t^2 - 18 a^2 b^2 t^2 + 9 b^4 t^2 + 16 a^2 t^2 δ + 14 a^4 t^2 δ - 32 a b t^2 δ + 16 b^2 t^2 δ -
       28 a^2 b^2 t^2 δ + 14 b^4 t^2 δ + 16 a^2 t^2 δ^2 + 5 a^4 t^2 δ^2 - 32 a b t^2 δ^2 + 16 b^2 t^2 δ^2 - 10 a^2 b^2 t^2 δ^2 +
       5 b^4 t^2 δ^2 - 18 a^2 t pB + 18 b^2 t pB + 16 a t δ pB - 10 a^2 t δ pB - 16 b t δ pB + 10 b^2 t δ pB + 9 pB^2)}, pB]]
```

$$\left\{\left\{p_B\to\frac{1}{90\,\delta}\,t\left(-80\,a\,\delta^2+80\,b\,\delta^2+3\,a^2\,\delta\,(9+5\,\delta)-3\,b^2\,\delta\,(9+5\,\delta)+3\,\left(81-25\,\delta^2\right)\sqrt{-\frac{\left(a^2-b^2\right)^2\,\delta^2}{-81+25\,\delta^2}}\right)\right\},\right.$$

$$\left.\left\{p_B\to\frac{1}{90\,\delta}\,t\left(-80\,a\,\delta^2+80\,b\,\delta^2+3\,a^2\,\delta\,(9+5\,\delta)-3\,b^2\,\delta\,(9+5\,\delta)+3\sqrt{-\frac{\left(a^2-b^2\right)^2\,\delta^2}{-81+25\,\delta^2}}\,\left(-81+25\,\delta^2\right)\right)\right\}\right\}$$

We can easily show that the latter outcome is negative. So, we use the former one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_B\to\frac{t\,(b-a)\,\delta\left((80\,\delta-3\,(a+b)\,(9+5\,\delta))+3\,(a+b)\,\sqrt{81-25\,\delta^2}\right)}{90\,\delta}\quad\text{(pb1)}$$

We rewrite the locations of the two endpoints: The right-hand endpoint in (i), the left-hand endpoints in (ii) (see below)

```
c1aR
c2aL
```

$$\left\{-\frac{1}{18}(a-b)\,t\,(9(a+b)+16\,\delta),\ \frac{8}{9}(-a+b)\,t\,\delta\right\}$$

$$\left\{\frac{1}{18}(a-b)\,t\,(-9(a+b)+2(-8+3a+3b)\,\delta),\ \frac{1}{9}(a-b)(-8+3a+3b)\,t\,\delta\right\}$$

We check the condition that the derived p_B (pb1) is below the p_B-element of the right-hand endpoints in (i).

$$\text{Simplify}\left[\text{Factor}\left[\frac{8}{9}(-a+b)\,t\,\delta-\frac{t\,(b-a)\,\delta\left((80\,\delta-3(a+b)(9+5\,\delta))+3(a+b)\sqrt{81-25\,\delta^2}\right)}{90\,\delta}\right]\right]$$

$$\frac{1}{30}(a-b)(a+b)\,t\left(-9-5\,\delta+\sqrt{81-25\,\delta^2}\right)$$

This is positive.

We also rewrite the location of the endpoint: The left endpoints in (i) (see below)

```
c1aL
```

$$\left\{\frac{1}{9}(a-b)(-8+5a+5b)\,t\,\delta,\ -\frac{1}{9}(-a+b)\,t\,(-8\,\delta+(a+b)(9+5\,\delta))\right\}$$

We check the condition that the derived p_B (pb1) is above the p_B-element of the left endpoints in (i).

$$\text{Simplify}\left[\text{Factor}\left[\frac{t\,(b-a)\,\delta\left((80\,\delta-3(a+b)(9+5\,\delta))+3(a+b)\sqrt{81-25\,\delta^2}\right)}{90\,\delta}-\left(-\frac{1}{9}(-a+b)\,t\,(-8\,\delta+(a+b)(9+5\,\delta))\right)\right]\right]$$

$$-\frac{1}{90}(a-b)(a+b)\,t\left(63+35\,\delta+3\sqrt{81-25\,\delta^2}\right)$$

This is positive.

For the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (i) is

$$\text{Simplify}\Big[\text{Expand}\Big[p_A\to\frac{1}{2(-9+3\,\delta c-5\,\delta f)}\Big((a-b)\,t\left(6\,\delta c-2\,\delta c^2+8\,\delta f+4\,\delta c\,\delta f+b\left(9-9\,\delta c+2\,\delta c^2+5\,\delta f-5\,\delta c\,\delta f\right)+a\left(9+2\,\delta c^2+5\,\delta f-\delta c\,(9+5\,\delta f)\right)\right)+(-9+3\,\delta c-10\,\delta f)\,p_B\Big)\ /.\ p_B\to\frac{1}{90\,\delta}t\,(b-a)\,\delta\left((80\,\delta-3(a+b)(9+5\,\delta))+3(a+b)\sqrt{81-25\,\delta^2}\right)\Big]\Big]$$

$$p_A\to-\frac{1}{180(9+5\,\delta)}(a-b)\,t\left(160\,\delta\,(9+5\,\delta)+3a\left(-50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)+5\,\delta\left(3+2\sqrt{81-25\,\delta^2}\right)\right)+3b\left(-50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)+5\,\delta\left(3+2\sqrt{81-25\,\delta^2}\right)\right)\right)$$

$$p_A\to\frac{(b-a)\,t}{180(9+5\,\delta)}\left(160\,\delta\,(9+5\,\delta)+3(a+b)\left((9+5\,\delta)(21-10\,\delta)+(9+10\,\delta)\sqrt{81-25\,\delta^2}\right)\right)$$

The jumping point of Firm A's reaction function in (i) is defined as c1ja

$$c1ja = \left\{\frac{(b-a)\,t}{180\,(9+5\,\delta)}\left(160\,\delta\,(9+5\,\delta)+3\,(a+b)\left((9+5\,\delta)\,(21-10\,\delta)+(9+10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{t\,(b-a)\,\delta\left((80\,\delta-3\,(a+b)\,(9+5\,\delta))+3\,(a+b)\sqrt{81-25\,\delta^2}\right)}{90\,\delta}\right\}$$

$$\left\{\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(160\,\delta\,(9+5\,\delta)+3\,(a+b)\left((21-10\,\delta)\,(9+5\,\delta)+(9+10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{1}{90}\,(-a+b)\,t\left(80\,\delta-3\,(a+b)\,(9+5\,\delta)+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right\}$$

Also, for the threshold value of p_B (pb1), the point of p_A-element in Firm A's reaction function in (ii) is

$$\text{Simplify}\left[\text{Expand}\left[p_A\to\left((9+3\,\delta c-10\,\delta f)\,p_B+(b-a)\,t\left((a+b)\left(9-\delta c^2-8\,\delta f+4\,\delta c\,\delta f\right)+2\left(3\,\delta c+\delta c^2+4\,\delta f-2\,\delta c\,\delta f\right)\right)\right)/\,(2\,(9+3\,\delta c-5\,\delta f))\ /.\ p_B\to\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((80\,\delta-3\,(a+b)\,(9+5\,\delta))+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right]\right]$$

$$p_A\to\frac{1}{180\,(-9+5\,\delta)}\,(a-b)\,t\left(160\,(9-5\,\delta)\,\delta+3\,a\left(50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)-5\,\delta\left(39+2\sqrt{81-25\,\delta^2}\right)\right)+3\,b\left(50\,\delta^2+9\left(21+\sqrt{81-25\,\delta^2}\right)-5\,\delta\left(39+2\sqrt{81-25\,\delta^2}\right)\right)\right)$$

$$p_A\to\frac{(b-a)\,t}{180\,(9-5\,\delta)}\left(160\,(9-5\,\delta)\,\delta+3\,(a+b)\left((9-5\,\delta)\,(21-10\,\delta)+(9-10\,\delta)\sqrt{81-25\,\delta^2}\right)\right)$$

The jumping point of Firm A's reaction function in (ii) is defined as c2ja

$$c2ja = \left\{\frac{(b-a)\,t}{180\,(9-5\,\delta)}\left(160\,(9-5\,\delta)\,\delta+3\,(a+b)\left((9-5\,\delta)\,(21-10\,\delta)+(9-10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{t\,(b-a)\,\delta\left((80\,\delta-3\,(a+b)\,(9+5\,\delta))+3\,(a+b)\sqrt{81-25\,\delta^2}\right)}{90\,\delta}\right\}$$

$$\left\{\frac{1}{180\,(9-5\,\delta)}\,(-a+b)\,t\left(160\,(9-5\,\delta)\,\delta+3\,(a+b)\left((21-10\,\delta)\,(9-5\,\delta)+(9-10\,\delta)\sqrt{81-25\,\delta^2}\right)\right),\ \frac{1}{90}\,(-a+b)\,t\left(80\,\delta-3\,(a+b)\,(9+5\,\delta)+3\,(a+b)\sqrt{81-25\,\delta^2}\right)\right\}$$

Next we show various examples of Firm A's 'true' reaction function for different values of *a*, *b*, *t*, δ, and *k*.

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 2
```

0

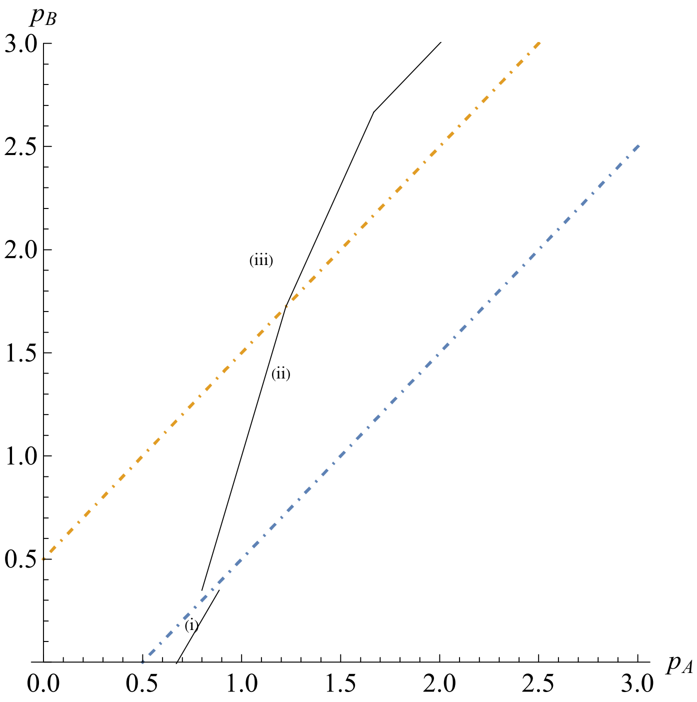
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}], Line[{c3daL, c3daR}],
    Text["(iii)", {1.1, 1.95}], Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
  PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
  AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0
b = 1
δ = 0.9
t = 1
k = 2
```

0

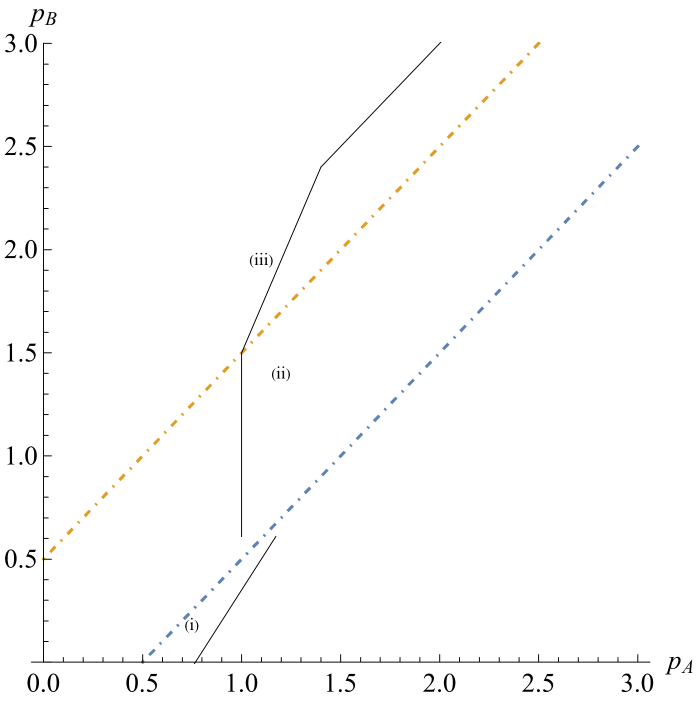
1

0.9

1

2

```
Plot[{x - (-3 a² t + 3 b² t - 4 a t δc + 3 a² t δc + 4 b t δc - 3 b² t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}], Line[{c3daL, c3daR}],
   Text["(iii)", {1.1, 1.95}], Text["(ii)", {1.2, 1.4}], Text["(i)", {0.75, 0.18}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
Clear[a, b, δ, t, k]
```

We now turn to Firm B's reaction function in the three cases. As before, we start with an example by setting *a=0*, *b=1*, *t=1*, and *δ=1/2*:

```
a = 0
b = 1
δ = 4 / 5
t = 1
k = 1
```

0

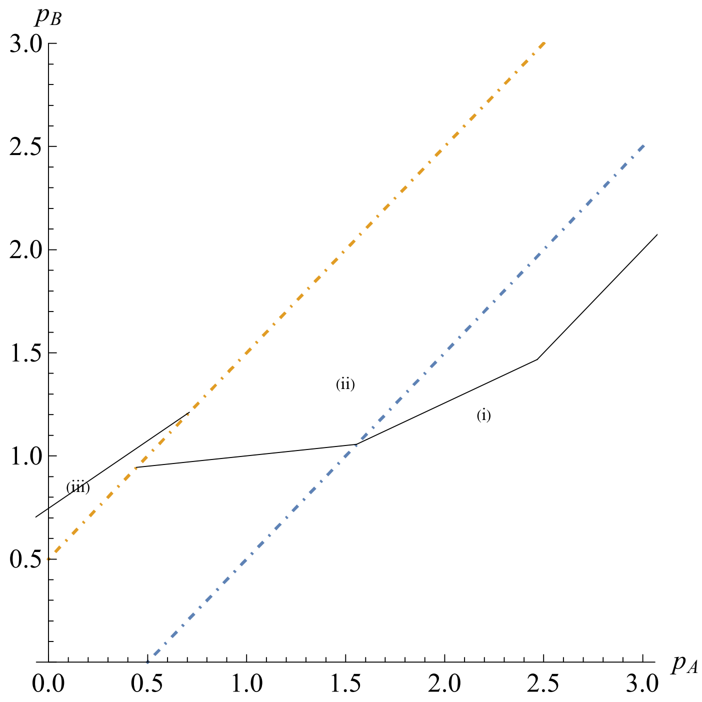
1

$\frac{4}{5}$

1

1

```
Plot[{x - (-3 a² t + 3 b² t - 4 a t δc + 3 a² t δc + 4 b t δc - 3 b² t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
 Epilog → {Line[{c1dbL, c1dbR}], Line[{c1bL, c1bR}], Line[{c2bL, c2bR}], Line[{c3bL, c3bR}],
   Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}]},
 PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```



We find that for some p_A, there are two local optimal prices for Firm B. We can show that the multiplicity of local optimal prices always appears.

```
Clear[a, b, δ, t, k]
```

To show the multiplicity, we check the locations of the two endpoints: The left-hand endpoint in (ii) (c2bL) and the left-hand and right-hand endpoints (c3bL and c3bR) in (iii) (see below)

```
c2bL
c3bL
c3bR
```

$$\left\{\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-2(2+3a+3b)\delta)\right\}$$

$$\left\{\frac{1}{9}(-a+b)\,t\,(-2(9+\delta)+(a+b)(9+5\delta)),\ -\frac{1}{9}(a-b)(-2+5a+5b)\,t\,\delta\right\}$$

$$\left\{\frac{8}{9}(-a+b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-16\delta)\right\}$$

First, we compare the elements of the left-hand endpoint in (ii) and the right-hand endpoint in (iii):

$$\text{Factor}\left[\frac{1}{9}\ (-a+b)\ (2+3a+3b)\ t\,\delta-\frac{8}{9}\ (-a+b)\ t\,\delta\right]$$

$$\text{Factor}\left[\frac{1}{18}\ (a-b)\ t\ (9\ (-2+a+b)-2\ (2+3a+3b)\ \delta)-\left(\frac{1}{18}\ (a-b)\ t\ (9\ (-2+a+b)-16\,\delta)\right)\right]$$

$$-\frac{1}{3}\ (a-b)\ (-2+a+b)\ t\,\delta$$

$$-\frac{1}{3}\ (a-b)\ (-2+a+b)\ t\,\delta$$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the right-hand endpoint in (iii) is located above the left-hand endpoint in (ii) as in the above Figure.

Second, we compare the elements of the left-hand endpoint in (ii) and the left-hand endpoint in (iii):

$$\text{Factor}\left[\frac{1}{9}\ (-a+b)\ (2+3a+3b)\ t\,\delta-\frac{1}{9}\ (-a+b)\ t\ (-2\ (9+\delta)+(a+b)\ (9+5\,\delta))\right]$$

$$\text{Factor}\left[\frac{1}{18}\ (a-b)\ t\ (9\ (-2+a+b)-2\ (2+3a+3b)\ \delta)-\left(-\frac{1}{9}\ (a-b)\ (-2+5a+5b)\ t\,\delta\right)\right]$$

$$\frac{1}{9}\ (a-b)\ (-2+a+b)\ t\ (9+2\,\delta)$$

$$\frac{1}{18}\ (a-b)\ (-2+a+b)\ t\ (9+4\,\delta)$$

The outcome means that for any a∈[0,1], b∈[0,1] (a≥b), t, and δ, the left-hand endpoint in (ii) is located above the left-hand endpoint in (iii) as in the above Figure.

We need to find the global optimal price of Firm B, p_B, when there are two local optima for a given p_A. There is a price p_A such that choosing the reaction function in (ii) and choosing the reaction function in (iii) are indifferent for Firm B. This p_A is the threshold in which choosing the reaction function in (ii) is preferred by Firm B if p_A is larger than the threshold p_A, otherwise, choosing the reaction function in (iii) is preferred by Firm B. We need to find the threshold value of p_A.

To check the threshold value of p_A for Firm B's reaction function, we derive the profits under cases (ii) and (iii).

The interior profit of firm B under case (ii) for p_A is

$$\text{Factor}\Big[p_B\ (1-z)+\delta f\ \frac{1}{9}\ (b-a)\ t\ \left(8-4a+a^2-4b+2ab+b^2-8z-2az-2bz+10z^2\right)\ /.$$
$$z\to\frac{3\ (p_B-p_A)}{2\ (b-a)\ t\ (3+\delta c)}+\frac{((a+b)\ (3-\delta c)+2\,\delta c)}{2\ (3+\delta c)}\ /.$$
$$\Big\{p_B\to\frac{1}{2\ (9+3\,\delta c-5\,\delta f)}\ \Big(-(a-b)\ t\ \big(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+$$
$$a\ \left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)+b\ \left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\big)+(9+3\,\delta c-10\,\delta f)\ p_A\Big)\Big\}\Big]$$

$$-\frac{1}{72\ (a-b)\ t\ (-9+5\,\delta)}\Big(-324a^2t^2+324a^3t^2-81a^4t^2+648abt^2-324a^2bt^2-324b^2t^2-324ab^2t^2+162a^2b^2t^2+324b^3t^2-$$
$$81b^4t^2-288a^2t^2\delta+144a^3t^2\delta-36a^4t^2\delta+576abt^2\delta-144a^2bt^2\delta-288b^2t^2\delta-144ab^2t^2\delta+$$
$$72a^2b^2t^2\delta+144b^3t^2\delta-36b^4t^2\delta+256a^2t^2\delta^2-192a^3t^2\delta^2+36a^4t^2\delta^2-512abt^2\delta^2+$$
$$192a^2bt^2\delta^2+256b^2t^2\delta^2+192ab^2t^2\delta^2-72a^2b^2t^2\delta^2-192b^3t^2\delta^2+36b^4t^2\delta^2+324atp_A-$$
$$162a^2tp_A-324btp_A+162b^2tp_A-216at\delta p_A+36a^2t\delta p_A+216bt\delta p_A-36b^2t\delta p_A-81p_A^2\Big)$$

The interior profit of firm B under case (iii) for p_B

$$\text{Factor}\left[p_B\ (1-z) + \delta f\left(\frac{1}{18}(b-a)\ t\ \left(-18a + a^2 - 18b + 2ab + b^2 + 36z + 10az + 10bz - 20z^2\right)\right) /. \ z \to \frac{-3a^2 t + 3b^2 t + 2a^2 t\,\delta c - 2b^2 t\,\delta c - 3p_A + 3p_B}{2(a-b)\ t\ (-3+\delta c)} /. \ p_B \to \frac{(9 - 3\delta c + 10\delta f)\ p_A}{2(9 - 3\delta c + 5\delta f)} + ((b-a)\ t\ (2(3-\delta c)(3-\delta c+3\delta f) - (a+b)((3-\delta c)(3-2\delta c) + 5(1-\delta c)\delta f)))/(2(9-3\delta c+5\delta f))\right]$$

$$-\frac{1}{8(a-b)\ t\ (9+5\delta)}\left(36a^2t^2 - 36a^3t^2 + 9a^4t^2 - 72abt^2 + 36a^2bt^2 + 36b^2t^2 + 36ab^2t^2 - 18a^2b^2t^2 - 36b^3t^2 + 9b^4t^2 + 72a^2t^2\delta - 56a^3t^2\delta + 14a^4t^2\delta - 144abt^2\delta + 56a^2bt^2\delta + 72b^2t^2\delta + 56ab^2t^2\delta - 28a^2b^2t^2\delta - 56b^3t^2\delta + 14b^4t^2\delta + 36a^2t^2\delta^2 - 20a^3t^2\delta^2 + 5a^4t^2\delta^2 - 72abt^2\delta^2 + 20a^2bt^2\delta^2 + 36b^2t^2\delta^2 + 20ab^2t^2\delta^2 - 10a^2b^2t^2\delta^2 - 20b^3t^2\delta^2 + 5b^4t^2\delta^2 - 36atp_A + 18a^2tp_A + 36btp_A - 18b^2tp_A - 4at\delta p_A + 10a^2t\delta p_A + 4bt\delta p_A - 10b^2t\delta p_A + 9p_A^2\right)$$

We derive the threshold value of p_B by finding p_B that equalizes the above two profits:

$$\text{FullSimplify}\Big[\text{Solve}\Big[\Big\{-\frac{1}{72(a-b)\ t\ (-9+5\delta)}\left(-324a^2t^2 + 324a^3t^2 - 81a^4t^2 + 648abt^2 - 324a^2bt^2 - 324b^2t^2 - 324ab^2t^2 + 162a^2b^2t^2 + 324b^3t^2 - 81b^4t^2 - 288a^2t^2\delta + 144a^3t^2\delta - 36a^4t^2\delta + 576abt^2\delta - 144a^2bt^2\delta - 288b^2t^2\delta - 144ab^2t^2\delta + 72a^2b^2t^2\delta + 144b^3t^2\delta - 36b^4t^2\delta + 256a^2t^2\delta^2 - 192a^3t^2\delta^2 + 36a^4t^2\delta^2 - 512abt^2\delta^2 + 192a^2bt^2\delta^2 + 256b^2t^2\delta^2 + 192ab^2t^2\delta^2 - 72a^2b^2t^2\delta^2 - 192b^3t^2\delta^2 + 36b^4t^2\delta^2 + 324atp_A - 162a^2tp_A - 324btp_A + 162b^2tp_A - 216at\delta p_A + 36a^2t\delta p_A + 216bt\delta p_A - 36b^2t\delta p_A - 81p_A^2\right) == -\frac{1}{8(a-b)\ t\ (9+5\delta)}\left(36a^2t^2 - 36a^3t^2 + 9a^4t^2 - 72abt^2 + 36a^2bt^2 + 36b^2t^2 + 36ab^2t^2 - 18a^2b^2t^2 - 36b^3t^2 + 9b^4t^2 + 72a^2t^2\delta - 56a^3t^2\delta + 14a^4t^2\delta - 144abt^2\delta + 56a^2bt^2\delta + 72b^2t^2\delta + 56ab^2t^2\delta - 28a^2b^2t^2\delta - 56b^3t^2\delta + 14b^4t^2\delta + 36a^2t^2\delta^2 - 20a^3t^2\delta^2 + 5a^4t^2\delta^2 - 72abt^2\delta^2 + 20a^2bt^2\delta^2 + 36b^2t^2\delta^2 + 20ab^2t^2\delta^2 - 10a^2b^2t^2\delta^2 - 20b^3t^2\delta^2 + 5b^4t^2\delta^2 - 36atp_A + 18a^2tp_A + 36btp_A - 18b^2tp_A - 4at\delta p_A + 10a^2t\delta p_A + 4bt\delta p_A - 10b^2t\delta p_A + 9p_A^2\right)\Big\}, p_A\Big]\Big]$$

$$\left\{\left\{p_A \to \frac{1}{90\delta}\ t\left(2a(27-25\delta)\delta - 3a^2\delta(9+5\delta) + 3b^2\delta(9+5\delta) + 2b\delta(-27+25\delta) + 3(81-25\delta^2)\sqrt{-\frac{(a-b)^2(-2+a+b)^2\delta^2}{-81+25\delta^2}}\right)\right\}, \left\{p_A \to \frac{1}{90\delta}\ t\left(2a(27-25\delta)\delta - 3a^2\delta(9+5\delta) + 3b^2\delta(9+5\delta) + 2b\delta(-27+25\delta) + 3\sqrt{-\frac{(a-b)^2(-2+a+b)^2\delta^2}{-81+25\delta^2}}\ (-81+25\delta^2)\right)\right\}\right\}$$

$$p_A \to \frac{1}{90\delta}\ t\ (b-a)\ \delta\left(\ (-54 + 50\delta + 3(a+b)(9+5\delta)) + 3(2-(a+b))\sqrt{(81-25\delta^2)}\ \right)$$

We can easily show that the latter outcome is negative. So, we use the former one.
We simplify the expression of the latter outcome, and obtain the following p_B:

$$p_A \to \frac{1}{90\delta}\ t\ (b-a)\ \delta\left(\ (-54 + 50\delta + 3(a+b)(9+5\delta)) + 3(2-(a+b))\sqrt{(81-25\delta^2)}\ \right)\ \text{(pa1)}$$

We rewrite the locations of the two endpoints: The right endpoint in (iii) and the left endpoint in (ii) (see below)

c3bR
c2bL

$$\left\{\frac{8}{9}(-a+b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-16\,\delta)\right\}$$

$$\left\{\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta,\ \frac{1}{18}(a-b)\,t\,(9(-2+a+b)-2(2+3a+3b)\,\delta)\right\}$$

We show that the derived p_A (pa1) is smaller than the p_A-element of the right-hand endpoints in (iii).

$$\text{Simplify}\left[\text{Factor}\left[\frac{8}{9}(-a+b)\,t\,\delta-\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3(a+b)(9+5\,\delta))+3(2-(a+b))\sqrt{(81-25\,\delta^2)}\right)\right]\right]$$

$$-\frac{1}{30}(a-b)(-2+a+b)\,t\left(-9-5\,\delta+\sqrt{81-25\,\delta^2}\right)$$

This is positive (note that $-9-5\,\delta+\sqrt{81-25\,\delta^2}$ is negative).

We show that the derived p_A (pa1) is larger than the p_A-element of the left-hand endpoints in (ii).

$$\text{Simplify}\left[\text{Factor}\left[\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3(a+b)(9+5\,\delta))+3(2-(a+b))\sqrt{(81-25\,\delta^2)}\right)-\frac{1}{9}(-a+b)(2+3a+3b)\,t\,\delta\right]\right]$$

$$\frac{1}{30}(a-b)(-2+a+b)\,t\left(-9+5\,\delta+\sqrt{81-25\,\delta^2}\right)$$

This is positive (note that $-9+5\,\delta+\sqrt{81-25\,\delta^2}$ is positive).

The reaction function of Firm B in (ii) is

$$p_B\to\frac{1}{2(9+3\,\delta c-5\,\delta f)}\left(-(a-b)\,t\left(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+a\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)$$

We substitute p_A into p_B:

$$\text{Simplify}\left[p_B\to\frac{1}{2(9+3\,\delta c-5\,\delta f)}\left(-(a-b)\,t\left(18+6\,\delta c-8\,\delta f+4\,\delta c\,\delta f+a\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)+b\left(-9+\delta c^2+8\,\delta f-4\,\delta c\,\delta f\right)\right)+(9+3\,\delta c-10\,\delta f)\,p_A\right)\ /.\ p_A\to\frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3(a+b)(9+5\,\delta))+3(2-(a+b))\sqrt{(81-25\,\delta^2)}\right)\right]$$

$$p_B\to\frac{1}{180(9-5\,\delta)}(-a+b)\,t\left(90(18-8\,\delta+a(-9+8\,\delta)+b(-9+8\,\delta))+(9-10\,\delta)\left(-54+50\,\delta+3(a+b)(9+5\,\delta)-3(-2+a+b)\sqrt{81-25\,\delta^2}\right)\right)$$

$$\text{c2jb} = \left\{\frac{1}{90\,\delta}\, t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\,\sqrt{(81-25\,\delta^2)}\right),\right.$$
$$\left.\frac{1}{180\,(9-5\,\delta)}\,(-a+b)\,t\left(90\,(18-8\,\delta+a\,(-9+8\,\delta)+b\,(-9+8\,\delta))+(9-10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

$$\left\{\frac{1}{90}\,(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\,\sqrt{81-25\,\delta^2}\right),\right.$$
$$\left.\frac{1}{180\,(9-5\,\delta)}\,(-a+b)\,t\left(90\,(18-8\,\delta+a\,(-9+8\,\delta)+b\,(-9+8\,\delta))+(9-10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

The reaction function of Firm B in (iii) is

$$p_B \to \frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))$$

We substitute p_A into p_B:

$$\text{Simplify}\left[p_B \to \frac{(9-3\,\delta c+10\,\delta f)\,p_A}{2\,(9-3\,\delta c+5\,\delta f)}+((b-a)\,t\,(2\,(3-\delta c)\,(3-\delta c+3\,\delta f)-(a+b)\,((3-\delta c)\,(3-2\,\delta c)+5\,(1-\delta c)\,\delta f)))\,/\,(2\,(9-3\,\delta c+5\,\delta f))\,/.\right.$$
$$\left.p_A \to \frac{1}{90\,\delta}\,t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\,\sqrt{(81-25\,\delta^2)}\right)\right]$$

$$p_B \to \frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)$$

$$\text{c3jb} = \left\{\frac{1}{90\,\delta}\, t\,(b-a)\,\delta\left((-54+50\,\delta+3\,(a+b)\,(9+5\,\delta))+3\,(2-(a+b))\,\sqrt{(81-25\,\delta^2)}\right),\right.$$
$$\left.\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

$$\left\{\frac{1}{90}\,(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\,\sqrt{81-25\,\delta^2}\right),\right.$$
$$\left.\frac{1}{180\,(9+5\,\delta)}\,(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\,\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

```
a = 0
b = 1
δ = 1 / 2
t = 1
k = 2
```

0

1

$\frac{1}{2}$

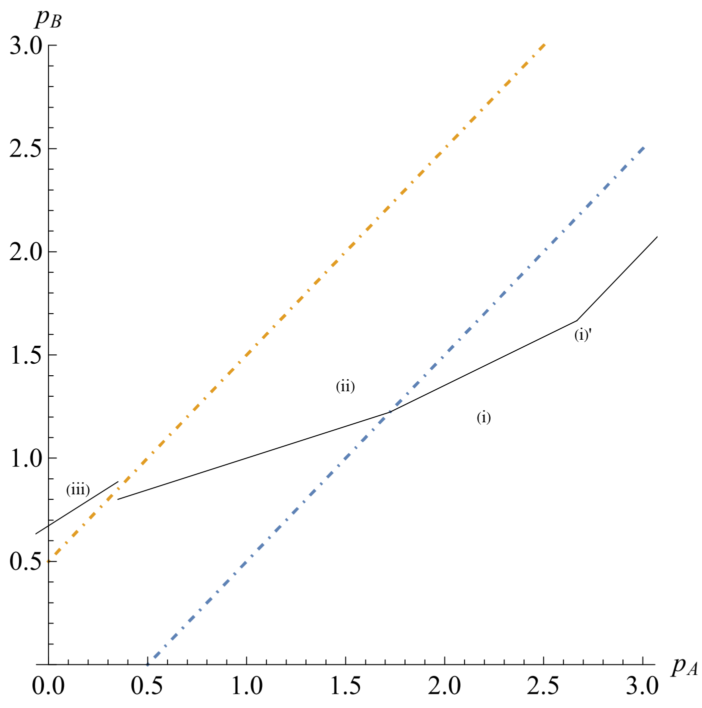
1

2

```
c1dbL
```

$\left\{\frac{1}{9}(a-b)(2+a+b)\,t\,(-9+4\,\delta),\ -\frac{2}{9}(a-b)\,t\,(9+(-7+a+b)\,\delta)\right\}$

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}], Line[{c2bR, c2jb}],
    Line[{c3jb, c3bL}], Text["(iii)", {0.15, 0.85}], Text["(ii)", {1.5, 1.35}],
    Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]}, PlotRange → {0, 3},
  LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"}, AspectRatio → 1, PlotStyle → DotDashed]
```



```
a = 0
b = 1
δ = 1/2
t = 1
k = 2
```

0

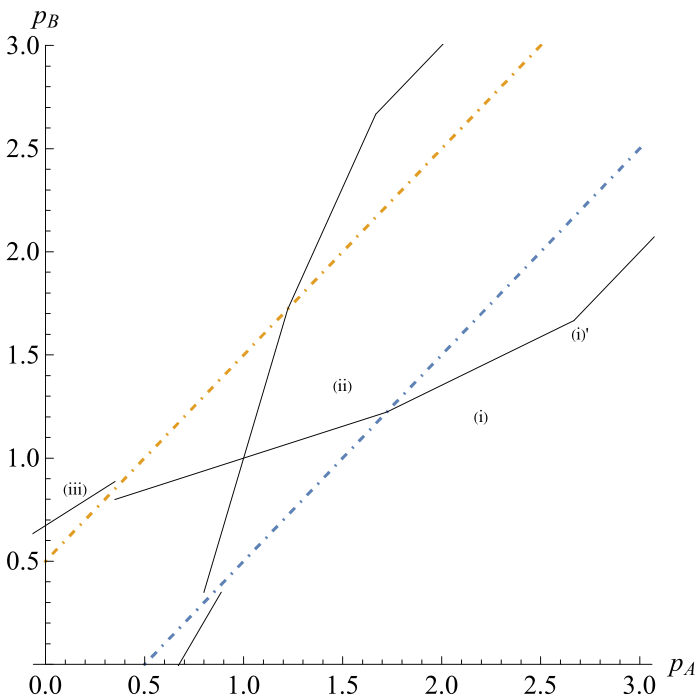
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 3},
  Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}],
    Line[{c3daL, c3daR}], , , Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}],
    Line[{c2bR, c2jb}], Line[{c3jb, c3bL}], , , Text["(iii)", {0.15, 0.85}],
    Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
  PlotRange → {0, 3}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
  AspectRatio → 1, PlotStyle → DotDashed]
```



```
a = 0.95
b = 1
δ = 1 / 2
t = 1
k = 2
```

0.95

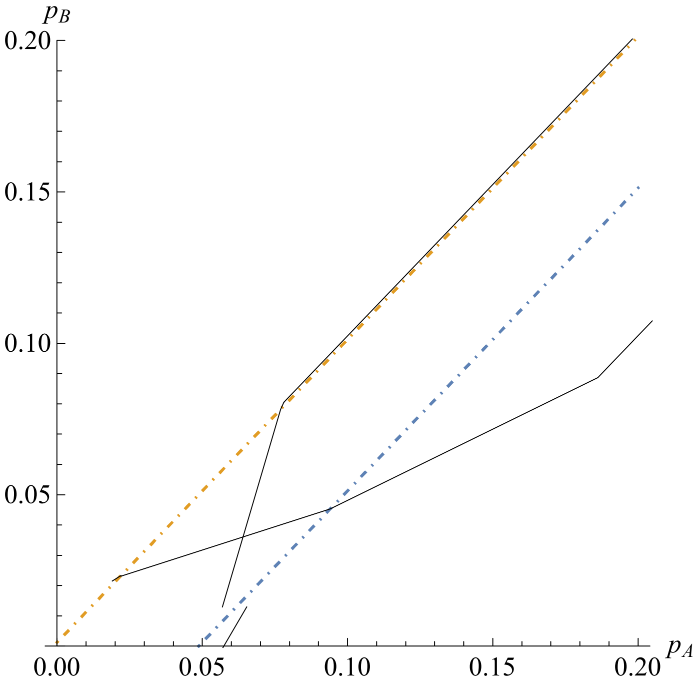
1

$\frac{1}{2}$

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
   x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 0.2},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}],
   Line[{c3daL, c3daR}], , , Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}],
   Line[{c2bR, c2jb}], Line[{c3jb, c3bL}], , , Text["(iii)", {0.15, 0.85}],
   Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
 PlotRange → {0, 0.2}, LabelStyle → (FontSize → 14), AxesLabel → {"p_A", "p_B"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
a = 0.95
b = 1
δ = 1
t = 1
k = 2
```

0.95

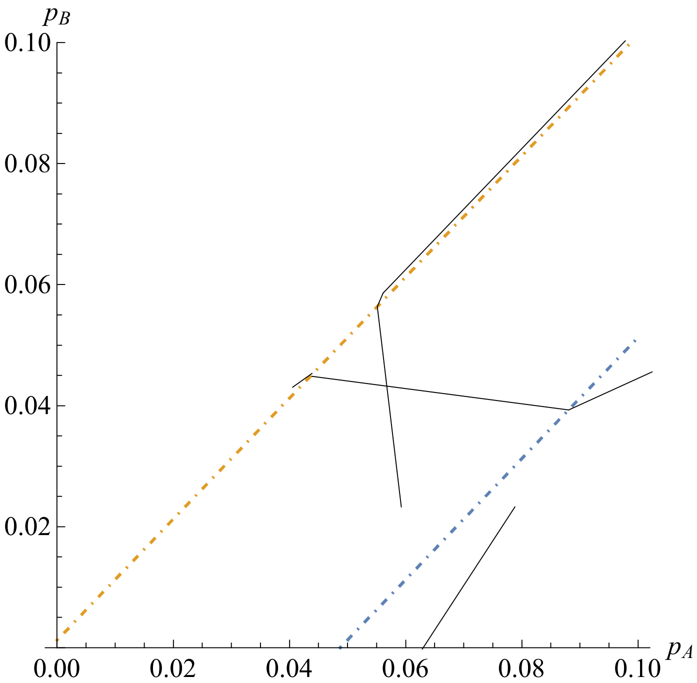
1

1

1

2

```
Plot[{x - (-3 a^2 t + 3 b^2 t - 4 a t δc + 3 a^2 t δc + 4 b t δc - 3 b^2 t δc) / 6,
  x + (b - a) t (6 + 3 a (-1 + δc) + 3 b (-1 + δc) - 2 δc) / 6}, {x, 0, 0.1},
 Epilog → {Line[{c1aL, c1ja}], Line[{c2ja, c2aR}], Line[{c3aL, c3aR}],
   Line[{c3daL, c3daR}], , , Line[{c1dbL, c1dbR}], Line[{c1bR, c1bL}],
   Line[{c2bR, c2jb}], Line[{c3jb, c3bL}], , , Text["(iii)", {0.15, 0.85}],
   Text["(ii)", {1.5, 1.35}], Text["(i)", {2.2, 1.2}], Text["(i)'", {2.7, 1.6}]},
 PlotRange → {0, 0.1}, LabelStyle → (FontSize → 14), AxesLabel → {"pA", "pB"},
 AspectRatio → 1, PlotStyle → DotDashed]
```




```
Clear[a, b, δ, t, k]
```

The above examples suggest that the pricing equilibrium is possible only in (Case ii). We now show that it is indeed the case for general values of (a, b, δ, t, k). To this end, we use the reaction functions corresponding to (Case ii), find the intersection of the two reactions functions, and show that the intersection point is always in (Case ii) for all values of δ.

```
FullSimplify[Solve[{pa ==
    ((9 + 3 δc - 10 δf) pb + (b - a) t ((a + b) (9 - δc^2 - 8 δf + 4 δc δf) + 2 (3 δc + δc^2 + 4 δf - 2 δc δf))) /
     (2 (9 + 3 δc - 5 δf)),
   pb == ((9 + 3 δc - 10 δf) pa + (b - a) t (18 + 6 δc - 8 δf + 4 δc δf + (a + b) (-9 + δc^2 + 8 δf - 4 δc δf))) /
     (2 (9 + 3 δc - 5 δf))}, {pa, pb}]]
```

$$\left\{\left\{pa \to -\frac{(a-b)\,t\,(-9\,(2+a+b)+4\,(3+2\,a+2\,b)\,\delta)}{-27+20\,\delta},\; pb \to \frac{(a-b)\,t\,(-9\,(-4+a+b)+4\,(-7+2\,a+2\,b)\,\delta)}{-27+20\,\delta}\right\}\right\}$$

We can easily show that the derived intersection is always in (Case ii).

We need to show that the intersection is stable even when we consider the reaction functions outside (Case ii).

From the result, if there is an intersection between the reaction functions of Firms A and B in (Case ii), the equilibrium p_B is the following:

$$\text{Eq } p_B : \frac{(a - b)\, t\, (-9\, (-4 + a + b) + 4\, (-7 + 2\, a + 2\, b)\, \delta)}{-27 + 20\, \delta}$$

We rewrite the jump point of Firm A's reaction function (c1ja):

```
c1ja
```

$$\left\{\frac{1}{180\, (9 + 5\, \delta)}\, (-a + b)\, t\, \left(160\, \delta\, (9 + 5\, \delta) + 3\, (a + b)\, \left((21 - 10\, \delta)\, (9 + 5\, \delta) + (9 + 10\, \delta)\, \sqrt{81 - 25\, \delta^2}\right)\right),\right.$$
$$\left.\frac{1}{90}\, (-a + b)\, t\, \left(80\, \delta - 3\, (a + b)\, (9 + 5\, \delta) + 3\, (a + b)\, \sqrt{81 - 25\, \delta^2}\right)\right\}$$

If the value of p_B in the jump point is smaller than the equilibrium p_B in (Case ii), the equilibrium point is stable. The difference between them is

$$\text{Simplify}\left[\text{Factor}\left[\frac{(a - b)\, t\, (-9\, (-4 + a + b) + 4\, (-7 + 2\, a + 2\, b)\, \delta)}{-27 + 20\, \delta} - \frac{1}{90}\, (-a + b)\, t\, \left(80\, \delta - 3\, (a + b)\, (9 + 5\, \delta) + 3\, (a + b)\, \sqrt{81 - 25\, \delta^2}\right)\right]\right]$$

$$\frac{1}{90\, (-27 + 20\, \delta)}\, (a - b)\, t\, \left(40\, \left(81 - 117\, \delta + 40\, \delta^2\right) + a\, \left(-300\, \delta^2 - 81\, \left(1 + \sqrt{81 - 25\, \delta^2}\right) + 15\, \delta\, \left(39 + 4\, \sqrt{81 - 25\, \delta^2}\right)\right) + b\, \left(-300\, \delta^2 - 81\, \left(1 + \sqrt{81 - 25\, \delta^2}\right) + 15\, \delta\, \left(39 + 4\, \sqrt{81 - 25\, \delta^2}\right)\right)\right)$$

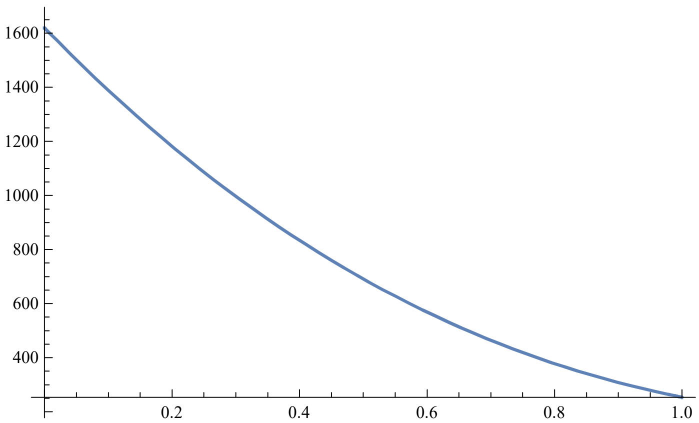
We arrange it, and obtain

$$\frac{1}{90\, (27 - 20\, \delta)}\, (b - a)\, t\, \left(40\, (9 - 5\, \delta)\, (9 - 8\, \delta) - 3\, (a + b)\, \left((9 - 5\, \delta)\, (3 - 20\, \delta) + (27 - 20\, \delta)\, \sqrt{81 - 25\, \delta^2}\right)\right)$$

The sign of this value just depends on δ and g≡a+b.
The value within the largest parentheses is linear in g(=a+b).
This value when g=0 is 40(9-5 δ)(9-8 δ), which positive.
We calculate this value when g=2, and obtain

$$2\, (9 - 5\, \delta)\, (171 - 100\, \delta) - 6\, (27 - 20\, \delta)\, \sqrt{81 - 25\, \delta^2}$$

$$\text{Plot}\left[2\, (9 - 5\, \delta)\, (171 - 100\, \delta) - 6\, (27 - 20\, \delta)\, \sqrt{81 - 25\, \delta^2},\, \{\delta, 0, 1\}\right]$$



From the intersection of the reaction functions in (Case ii), the equilibrium p_A is the following:

$$\text{Eq pa} : -\frac{(a - b)\, t\, (-9\, (2 + a + b) + 4\, (3 + 2\, a + 2\, b)\, \delta)}{-27 + 20\, \delta}$$

We rewrite the jump point of Firm B's reaction function (c3jb):

c3jb

$$\left\{\frac{1}{90}(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\sqrt{81-25\,\delta^2}\right),\right.$$
$$\left.\frac{1}{180\,(9+5\,\delta)}(-a+b)\,t\left(90\,(18\,(1+\delta)-(a+b)\,(9+5\,\delta))+(9+10\,\delta)\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)-3\,(-2+a+b)\sqrt{81-25\,\delta^2}\right)\right)\right\}$$

If the value of p_A in the jump point is smaller than the equilibrium p_A in (Case ii), the equilibrium point is stable. The difference between them is

$$\mathbf{Simplify}\left[\mathbf{Factor}\left[-\frac{(a-b)\,t\,(-9\,(2+a+b)+4\,(3+2\,a+2\,b)\,\delta)}{-27+20\,\delta}-\frac{1}{90}(-a+b)\,t\left(-54+50\,\delta+3\,(a+b)\,(9+5\,\delta)+3\,(2-a-b)\sqrt{81-25\,\delta^2}\right)\right]\right]$$

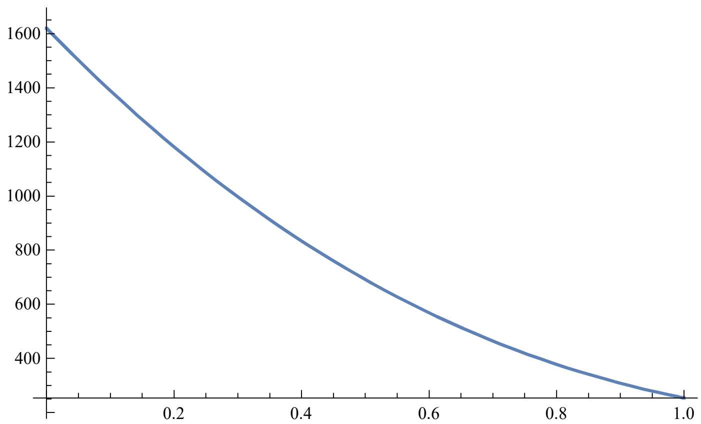
$$\frac{1}{90\,(-27+20\,\delta)}(a-b)\,t\left(2\left(500\,\delta^2-81\left(-19+\sqrt{81-25\,\delta^2}\right)+15\,\delta\left(-117+4\sqrt{81-25\,\delta^2}\right)\right)+a\left(300\,\delta^2+81\left(1+\sqrt{81-25\,\delta^2}\right)-15\,\delta\left(39+4\sqrt{81-25\,\delta^2}\right)\right)+b\left(300\,\delta^2+81\left(1+\sqrt{81-25\,\delta^2}\right)-15\,\delta\left(39+4\sqrt{81-25\,\delta^2}\right)\right)\right)$$

We arrange it, and obtain

$$\frac{(b-a)\,t}{90\,(27-20\,\delta)}\left(2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\sqrt{81-25\,\delta^2}\right)+3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and g≡a+b.
The value within the largest parentheses is linear in g(=a+b).
This value when g=0 is $2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\sqrt{81-25\,\delta^2}\right)$, which positive.
We calculate this value when g=2, and obtain 40 (9-5 δ) (9-8 δ), which is positive.

$$\mathbf{Plot}\left[\left\{2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\sqrt{81-25\,\delta^2}\right)\right\},\ \{\delta,\ 0,\ 1\}\right]$$



We have shown that the equilibrium is given by $a=0$, $b=1$ for all values of δ. The rest of Proposition 2 follows by substituting a = 0, b = 1, $\delta_f=\delta$ and $\delta_c=0$ into relevant prices and the locations of marginal consumers in the two periods.

The following discussion is related to Proposition 3.

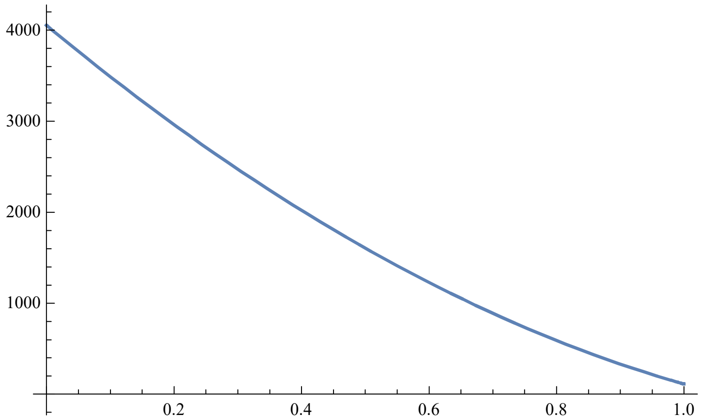
Here, we expand the range of g(=a+b) from -1 to 3.
Check the equilibrium locations in which the locations of the firms are restricted within the range [-1/2,3/2].

$$\frac{1}{90\,(27-20\,\delta)}\,(b-a)\,t\left(40\,(9-5\,\delta)\,(9-8\,\delta)-3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\,\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and “a+b”.
Here we define g≡a+b. The value within the largest parentheses is linear in g(=a+b).

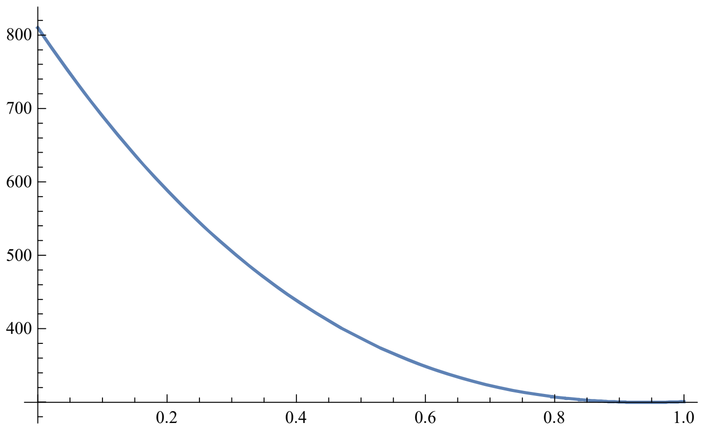
This value when g=-1is

```
Plot[40 (9 - 5 δ) (9 - 8 δ) - 3 * (-1) * ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ^2)), {δ, 0, 1}]
```



We calculate this value when g=3, and obtain

```
Plot[40 (9 - 5 δ) (9 - 8 δ) - 3 *3* ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ^2)), {δ, 0, 1}]
```

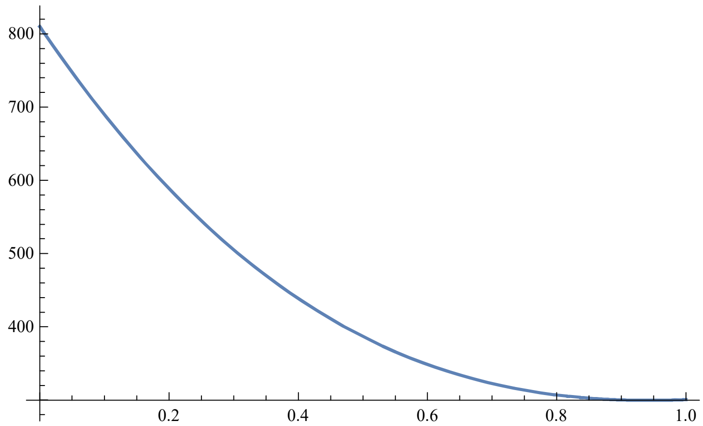


$$\frac{(b-a)\,t}{90\,(27-20\,\delta)}\left(2\left((9-5\,\delta)\,(171-100\,\delta)-(81-60\,\delta)\,\sqrt{81-25\,\delta^2}\right)+\right.$$
$$\left.3\,(a+b)\left((9-5\,\delta)\,(3-20\,\delta)+(27-20\,\delta)\,\sqrt{81-25\,\delta^2}\right)\right)$$

The sign of this value just depends on δ and “a+b”.
Here we define g≡a+b. The value within the largest parentheses is linear in g(=a+b).

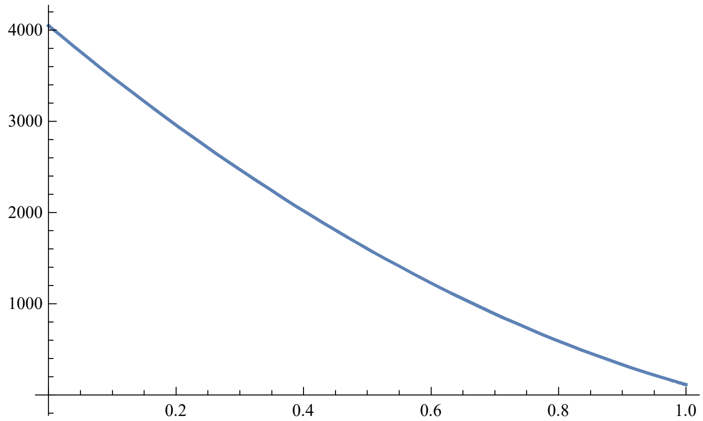
This value when g=-1 is

```
Plot[{2 ((9 - 5 δ) (171 - 100 δ) - (81 - 60 δ) √(81 - 25 δ^2)) +
   3 * (-1) ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ^2))}, {δ, 0, 1}]
```



We calculate this value when g=3 is

```
Plot[{2 ((9 - 5 δ) (171 - 100 δ) - (81 - 60 δ) √(81 - 25 δ^2)) +
   3 * 3 ((9 - 5 δ) (3 - 20 δ) + (27 - 20 δ) √(81 - 25 δ^2))}, {δ, 0, 1}]
```



Check the equilibrium locations in which the locations of the firms are restricted within the range [-1/2,3/2].

The following is the candidate locations of Firms A and B

$$a: -\frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)}$$

$$b: 1 + \frac{81 - 99\,\delta f + 20\,\delta f^2}{12\,(27 + 9\,\delta f - 20\,\delta f^2)}$$

The profit of Firm A in case (ii)

$$\text{Factor}\Big[pa\, z + \delta f \frac{1}{9} (b - a)\, t\, \left(2 + 2 a + a^2 + 2 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2\right) /. \; z \to \frac{3\,(pb - pa)}{2\,(b - a)\, t\,(3 + \delta c)} + \frac{((a + b)\,(3 - \delta c) + 2\,\delta c)}{2\,(3 + \delta c)} /. \Big\{pa \to \frac{1}{81 + 27\,\delta c - 60\,\delta f}\,(a - b)\, t \left(-27\,(2 + a + b) - 54\,\delta c + 3\,(-4 + a + b)\,\delta c^2 - 4\,(-9 + 3 a\,(-2 + \delta c) + 3 b\,(-2 + \delta c) - 8\,\delta c)\,\delta f\right),\; pb \to -\frac{1}{81 + 27\,\delta c - 60\,\delta f}\,(a - b)\, t \left(-27\,(-4 + a + b) + 54\,\delta c + 3\,(2 + a + b)\,\delta c^2 - 4\,(21 + 3 a\,(-2 + \delta c) + 3 b\,(-2 + \delta c) + 2\,\delta c)\,\delta f\right)\Big\} /. \{\delta c \to 0\}\Big]$$

$$-\frac{1}{18\,(-27 + 20\,\delta f)^2}\,(a - b)\, t \left(2916 + 2916 a + 729 a^2 + 2916 b + 1458 a b + 729 b^2 - 3888\,\delta f - 2592 a\,\delta f + 405 a^2\,\delta f - 2592 b\,\delta f + 810 a b\,\delta f + 405 b^2\,\delta f + 864\,\delta f^2 - 1008 a\,\delta f^2 - 1656 a^2\,\delta f^2 - 1008 b\,\delta f^2 - 3312 a b\,\delta f^2 - 1656 b^2\,\delta f^2 + 320\,\delta f^3 + 960 a\,\delta f^3 + 720 a^2\,\delta f^3 + 960 b\,\delta f^3 + 1440 a b\,\delta f^3 + 720 b^2\,\delta f^3\right)$$

The first-order derivative of Firm A’s profit with respect to *a* is

```
Factor[D[%, a]]
```

$$-\frac{1}{18\,(-27 + 20\,\delta f)^2}\, t \left(2916 + 5832 a + 2187 a^2 + 1458 a b - 729 b^2 - 3888\,\delta f - 5184 a\,\delta f + 1215 a^2\,\delta f + 810 a b\,\delta f - 405 b^2\,\delta f + 864\,\delta f^2 - 2016 a\,\delta f^2 - 4968 a^2\,\delta f^2 - 3312 a b\,\delta f^2 + 1656 b^2\,\delta f^2 + 320\,\delta f^3 + 1920 a\,\delta f^3 + 2160 a^2\,\delta f^3 + 1440 a b\,\delta f^3 - 720 b^2\,\delta f^3\right)$$

We substitute the candidate location of Firm B into the first-order derivative:

$$\text{Factor}\left[\% /. \; b \to 1 + \frac{81 - 99\,\delta f + 20\,\delta f^2}{12\left(27 + 9\,\delta f - 20\,\delta f^2\right)}\right]$$

$$-\Big(\Big(t\left(-81 - 324 a + 99\,\delta f - 108 a\,\delta f - 20\,\delta f^2 + 240 a\,\delta f^2\right)\left(-255879 - 78732 a + 65610\,\delta f - 69984 a\,\delta f + 420957\,\delta f^2 + 222588 a\,\delta f^2 - 148068\,\delta f^3 + 14256 a\,\delta f^3 - 194000\,\delta f^4 - 158400 a\,\delta f^4 + 91200\,\delta f^5 + 57600 a\,\delta f^5\right)\Big) \Big/ \left(288\,(-27 + 20\,\delta f)^2\left(-27 - 9\,\delta f + 20\,\delta f^2\right)^2\right)\Big)$$

We derive *a* which makes the above first-order derivative equal to zero.

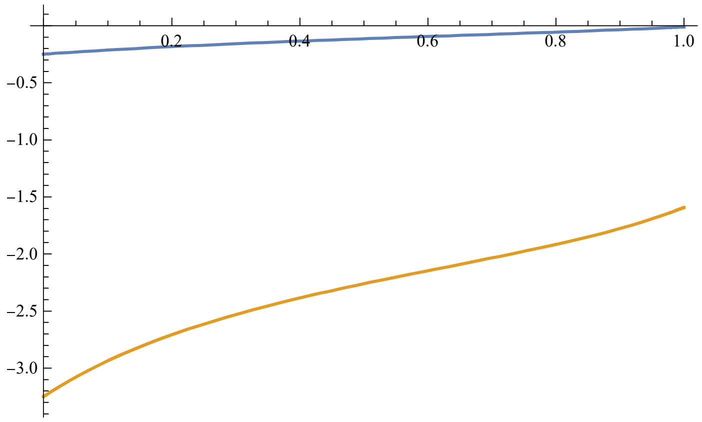
```
Simplify[Solve[% == 0, a]]
```

$$\left\{\left\{a \to \frac{81 - 99\,\delta f + 20\,\delta f^2}{12\left(-27 - 9\,\delta f + 20\,\delta f^2\right)}\right\}, \left\{a \to \left(255879 - 65610\,\delta f - 420957\,\delta f^2 + 148068\,\delta f^3 + 194000\,\delta f^4 - 91200\,\delta f^5\right) \Big/ \left(36\left(-2187 - 1944\,\delta f + 6183\,\delta f^2 + 396\,\delta f^3 - 4400\,\delta f^4 + 1600\,\delta f^5\right)\right)\right\}\right\}$$

The former *a* coincides with the candidate location of Firm A.

To check the sign of the first-order condition, we draw the values of *a* just derived above.

```
Plot[{(81 - 99 δf + 20 δf^2)/(12 (-27 - 9 δf + 20 δf^2)),
  (255 879 - 65 610 δf - 420 957 δf^2 + 148 068 δf^3 + 194 000 δf^4 - 91 200 δf^5) /
   (36 (-2187 - 1944 δf + 6183 δf^2 + 396 δf^3 - 4400 δf^4 + 1600 δf^5))}, {δf, 0, 1}]
```



The first-order derivative with $b = 1 + \frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$ is positive if $a < \frac{81-99\,\delta f+20\,\delta f^2}{12\,(-27-9\,\delta f+20\,\delta f^2)}$, otherwise it is negative. Therefore, $a = -\frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$ is the best location for Firm A given that Firm B chooses $b = 1 + \frac{81-99\,\delta f+20\,\delta f^2}{12\,(27+9\,\delta f-20\,\delta f^2)}$.

The profit of Firm B in (Case ii)

```
Factor[pb (1 - z) + δf 1/9 (b - a) t (8 - 4 a + a^2 - 4 b + 2 a b + b^2 - 8 z - 2 a z - 2 b z + 10 z^2) /.
   z → (3 (pb - pa))/(2 (b - a) t (3 + δc)) + ((a + b) (3 - δc) + 2 δc)/(2 (3 + δc)) /. {pa → 1/(81 + 27 δc - 60 δf) (a - b) t
      (-27 (2 + a + b) - 54 δc + 3 (-4 + a + b) δc^2 - 4 (-9 + 3 a (-2 + δc) + 3 b (-2 + δc) - 8 δc) δf),
   pb → -1/(81 + 27 δc - 60 δf) (a - b) t (-27 (-4 + a + b) + 54 δc + 3 (2 + a + b) δc^2 -
       4 (21 + 3 a (-2 + δc) + 3 b (-2 + δc) + 2 δc) δf)} /. {δc → 0}]
```

$$-\frac{1}{18\,(-27+20\,\delta f)^2}(a-b)\,t\,\left(11664-5832\,a+729\,a^2-5832\,b+1458\,a\,b+729\,b^2-7452\,\delta f+972\,a\,\delta f+405\,a^2\,\delta f+972\,b\,\delta f+810\,a\,b\,\delta f+405\,b^2\,\delta f-7776\,\delta f^2+7632\,a\,\delta f^2-1656\,a^2\,\delta f^2+7632\,b\,\delta f^2-3312\,a\,b\,\delta f^2-1656\,b^2\,\delta f^2+5120\,\delta f^3-3840\,a\,\delta f^3+720\,a^2\,\delta f^3-3840\,b\,\delta f^3+1440\,a\,b\,\delta f^3+720\,b^2\,\delta f^3\right)$$

The first-order derivative of Firm B's profit with respect to b is

```
Factor[D[%, b]]
```

$$-\frac{1}{18\,(-27+20\,\delta f)^2}t\,\left(-11664+729\,a^2+11664\,b-1458\,a\,b-2187\,b^2+7452\,\delta f+405\,a^2\,\delta f-1944\,b\,\delta f-810\,a\,b\,\delta f-1215\,b^2\,\delta f+7776\,\delta f^2-1656\,a^2\,\delta f^2-15264\,b\,\delta f^2+3312\,a\,b\,\delta f^2+4968\,b^2\,\delta f^2-5120\,\delta f^3+720\,a^2\,\delta f^3+7680\,b\,\delta f^3-1440\,a\,b\,\delta f^3-2160\,b^2\,\delta f^3\right)$$

We substitute the candidate location of Firm A into the first-order derivative:

```
Factor[% /. a → -(81 - 99 δf + 20 δf^2)/(12 (27 + 9 δf - 20 δf^2))]
```

$$\left(t\left(405-324\,b+9\,\delta f-108\,b\,\delta f-220\,\delta f^2+240\,b\,\delta f^2\right)\left(334611-78732\,b+4374\,\delta f-69984\,b\,\delta f-643545\,\delta f^2+222588\,b\,\delta f^2+133812\,\delta f^3+14256\,b\,\delta f^3+352400\,\delta f^4-158400\,b\,\delta f^4-148800\,\delta f^5+57600\,b\,\delta f^5\right)\right)\Big/\left(288\left(-27+20\,\delta f\right)^2\left(-27-9\,\delta f+20\,\delta f^2\right)^2\right)$$

We derive a which makes the above first-order derivative equal to zero:

```
Simplify[Solve[% == 0, b]]
```

$$\left\{\left\{b\to\frac{405+9\,\delta f-220\,\delta f^2}{324+108\,\delta f-240\,\delta f^2}\right\},\left\{b\to-\left(\left(334611+4374\,\delta f-643545\,\delta f^2+133812\,\delta f^3+352400\,\delta f^4-148800\,\delta f^5\right)\Big/\left(36\left(-2187-1944\,\delta f+6183\,\delta f^2+396\,\delta f^3-4400\,\delta f^4+1600\,\delta f^5\right)\right)\right)\right\}\right\}$$

We rewrite the former *b*:

```
Factor[(405 + 9 δf - 220 δf^2)/(324 + 108 δf - 240 δf^2) - 1]
```

$$-\frac{81-99\,\delta f+20\,\delta f^2}{12\left(-27-9\,\delta f+20\,\delta f^2\right)}$$

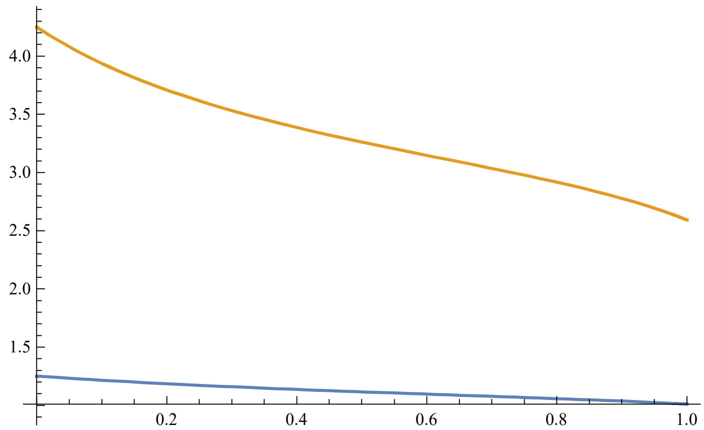
The former *b* is

$$b:1+\frac{81-99\,\delta f+20\,\delta f^2}{12\left(27+9\,\delta f-20\,\delta f^2\right)}$$

The former *b* coincides with the candidate location of Firm B.

To check the sign of the first-order derivative, we draw the values of *b* just derived above.

```
Plot[{1 + (81 - 99 δf + 20 δf^2)/(12 (27 + 9 δf - 20 δf^2)),
  -((334611 + 4374 δf - 643545 δf^2 + 133812 δf^3 + 352400 δf^4 - 148800 δf^5) /
     (36 (-2187 - 1944 δf + 6183 δf^2 + 396 δf^3 - 4400 δf^4 + 1600 δf^5)))}, {δf, 0, 1}]
```



The first-order derivative with a= $-\frac{81-99\,\delta f+20\,\delta f^2}{12\left(27+9\,\delta f-20\,\delta f^2\right)}$ is positive if b $<1+\frac{81-99\,\delta f+20\,\delta f^2}{12\left(27+9\,\delta f-20\,\delta f^2\right)}$, otherwise it is negative. Therefore, b= $1+\frac{81-99\,\delta f+20\,\delta f^2}{12\left(27+9\,\delta f-20\,\delta f^2\right)}$ is the optimal location choice for Firm B given that Firm A chooses $a=-\frac{81-99\,\delta f+20\,\delta f^2}{12\left(27+9\,\delta f-20\,\delta f^2\right)}$.