Hiroshi Kitamura, Noriaki Matsushima, and Misato Sato, (2023). How Does Downstream Firms' Efficiency Affect Exclusive Supply Agreements? *Review of Industrial Organization*.

# Supplementary Appendix

Section A1 explains the mathematical proofs related to the paragraph right after Remark 1 and footnote 15 in Section 4.3. Section A2 explains the result under Cournot competition referred in the first paragraph of Section 4.4. Section A3 explains the result referred in the third paragraph of Section 4.4. Section A4 explains the result referred in the second last paragraph of Section 5.1. Section A5 explains the result referred in the second paragragph of Section 4.4.

## A1 Demand system

We consider two demand systems: (i) perfectly inelastic demand with the size $\overline{Q}$ and the willingness to pay $v(> c)$, (ii) constant elasticity of demand $q = \alpha p^{-\varepsilon}$, where $\alpha > 0$ is a positive constant and $\varepsilon > 1$ is the constant elasticity of demand.

**Perfectly inelastic demand** First, we consider the case of exclusion. $D_I$ sets $p^a = v$ whenever it is profitable. Anticipating this pricing, $U$ sets $w^a = v$ and the consumer demand and input demand are $\overline{Q}$. The joint profit of $D_I$ and $U$ becomes $\pi_I^a + \pi_U^a = 0 + (v - c)\overline{Q} = (v - c)\overline{Q}$.

Second, we consider the case of entry by $D_E$. If $w \leq v$, the retail price by $D_E$ is $p^r = w$. If $w \in (v, v/k]$, $D_E$'s retail price is $p^r = v$. Anticipating this pricing, $U$ sets $w = v/k$ and the consumer demand and input demand are respectively $\overline{Q}$ and $k\overline{Q}$. The joint profit of $D_I$ and $U$ becomes $\pi_I^r + \pi_U^r = 0 + (v/k - c)k\overline{Q} = (v - c/k)\overline{Q}$, which is larger than that in the case of exclusion. Thus, condition (5) in the main text does not hold; exclusive dealing is not achievable under perfectly inelastic demand.

Under perfectly inelastic demand, $U$ can extract all surplus in the market under both the exclusion and entry cases. Interestingly, $U$ sets its wholesale price higher than each consumer's willingness to pay $v$ under the entry case, which implies that the efficient downstream firm $D_E$ is more useful for $U$. Thus, $D_I$ cannot deter the entry of $D_E$ through exclusive supply contracts under perfectly inelastic demand.

**Constant-elasticity demand** First, we consider the case of exclusion. Given $w$, $D_I$ solves

$$\max_p\ (p-w)\alpha p^{-\varepsilon}.$$

The resulting price is $p^a(w)=\varepsilon w/(\varepsilon-1)$. Anticipating this pricing, $U$ solves

$$\max_w\ (w-c)\alpha\left(\frac{\varepsilon w}{\varepsilon-1}\right)^{-\varepsilon}.$$

The resulting wholesale price is $w^a=\varepsilon c/(\varepsilon-1)$. The joint profit of $D_I$ and $U$ becomes

$$\pi_I^a+\pi_U^a=\frac{\alpha c^{-\varepsilon+1}(2\varepsilon-1)}{(\varepsilon-1)^2}\left(\frac{\varepsilon^2}{(\varepsilon-1)^2}\right)^{-\varepsilon}.$$

Second, we consider the case of entry by $D_E$. We consider two cases: (i) $k\geq(\varepsilon-1)/\varepsilon$; (ii) $k\leq(\varepsilon-1)/\varepsilon$. In Case (i), given $w$, $D_E$ sets $p^{r(i)}(w)=w$, which is the marginal cost of $D_E$. In Case (ii), given $w$, $D_E$ sets $p^{r(ii)}(w)=k\varepsilon w/(\varepsilon-1)$, which is the unconstrained monopoly price.

In Case (i), anticipating $D_E$'s pricing, $p^{r(i)}(w)=w$, $U$ solves

$$\max_w\ (w-c)k\alpha w^{-\varepsilon}.$$

The resulting wholesale price is $w^{r(i)}=\varepsilon c/(\varepsilon-1)$. The joint profit of $D_I$ and $U$ becomes

$$\pi_I^{r(i)}+\pi_U^{r(i)}=\frac{\alpha c^{-\varepsilon+1}k}{(\varepsilon-1)}\left(\frac{\varepsilon}{\varepsilon-1}\right)^{-\varepsilon}.$$

In Case (ii), anticipating $D_E$'s pricing, $p^{r(ii)}(w)=\varepsilon w/(\varepsilon-1)$, $U$ solves

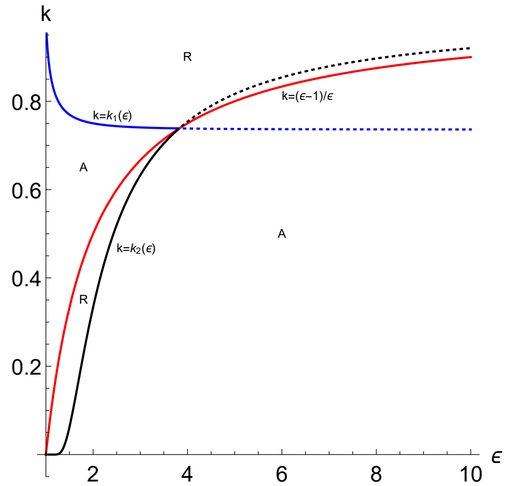
$$\max_w\ (w-c)k\alpha\left(\frac{\varepsilon w}{\varepsilon-1}\right)^{-\varepsilon}.$$

The resulting wholesale price is $w^{r(ii)}=\varepsilon c/(\varepsilon-1)$. The joint profit of $D_I$ and $U$ becomes

$$\pi_I^{r(ii)}+\pi_U^{r(ii)}=\frac{\alpha c^{-\varepsilon+1}k^{-\varepsilon+1}}{(\varepsilon-1)}\left(\frac{\varepsilon}{\varepsilon-1}\right)^{-2\varepsilon}.$$

Condition (5) in the main text holds ($A$: Accept; $R$: Reject in the figure):

$$\text{if } \frac{\varepsilon-1}{\varepsilon}\leq k,\quad k<\frac{2\varepsilon-1}{\varepsilon}\left(\frac{\varepsilon}{\varepsilon-1}\right)^{1-\varepsilon}\equiv k_1(\varepsilon);$$
$$\text{if } k\leq\frac{\varepsilon-1}{\varepsilon},\quad k\leq\left(\frac{2\varepsilon-1}{\varepsilon-1}\right)^{\frac{1}{1-\varepsilon}}\equiv k_2(\varepsilon).$$




**The result of the constant-elasticity demand case**

## A2 Cournot model

This section considers the existence of anticompetitive exclusive dealings to deter the socially efficient entry of $D_E$ when downstream firms compete in quantity. In this section, we use linear demand; that is, the inverse demand for the final product $P(Q) = \alpha - \beta Q$, where $Q$ is the output of the final product supplied by downstream firms and where $\alpha > c$ and $\beta > 0$. The following result shows that exclusion is possible not only under price competition but under Cournot competition as well.

**Result under Cournot competition** *Suppose that the downstream firms compete in quantity. Then, there can be an exclusion equilibrium as $D_E$ becomes efficient (that is, $k < \hat{k}$), where $\hat{k} \simeq 0.921543$. More precisely,*

1. *for $\hat{k} \leq k < 1$, entry is a unique equilibrium outcome, and*

2. *for $0 < k < \hat{k}$, exclusion is a unique equilibrium outcome if $U$ is sufficiently efficient, that is, $0 \leq c < \overline{C}(k)$ and*

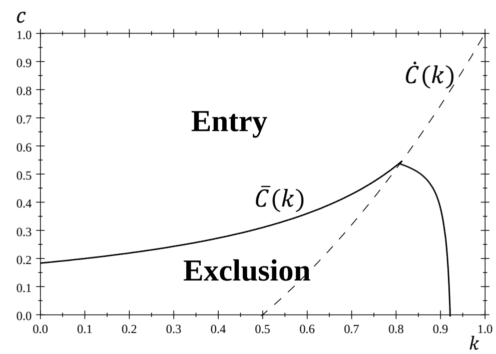
$$\overline{C}(k) = \begin{cases} \dfrac{a\left(2k^3 + 3k^2 + 3k - 7 + \sqrt{27(1-k)^2(-4k^4 - 12k^3 + 31k^2 - 26k + 10)}\right)}{(2k-1)(14k-13)(k^2-k+1)} & \textit{if } c < \dot{C}(k), \\ \frac{(\sqrt{6}-2)\alpha}{\sqrt{6}-2k} & \textit{if } c \geq \dot{C}(k), \end{cases} \tag{1}$$

*where*

$$\dot{C}(k) = \frac{a\left(k^2 - k + 1 - \sqrt{3(1-k)^2(k^2-k+1)}\right)}{(2-k)(k^2-k+1)}. \tag{2}$$

*Note that* $\partial\dot{C}(k)/\partial k > 0$, $\dot{C}(k) \to 0$ *as* $k \to 1/2$, *and* $\dot{C}(k) \to 1$ *as* $k \to 1$ *and that* $\partial\overline{C}(k)/\partial k < (\geq) 0$ *for* $k > (\leq) \dot{K}(c_A)$, $\overline{C}(k) \to a(3-\sqrt{6})/3 \simeq 0.1835a$ *as* $k \to 0$, *and* $\overline{C}(k) \to 0$ *as* $k \to \hat{k}$.

The following figure summarizes the above result.




**The result under Cournot competition**

**Proof of the Cournot model** We consider each of the possible subgames after Stage 1. Note that when $U$ rejects the exclusive supply offer, $D_E$ enters the downstream market and $U$ deals with $D_I$ and $D_E$. Given $w$, the quantities supplied by the downstream firms are given as

$$q_I^r(w) = \begin{cases} \dfrac{a-(2-k)w}{3b}, & \text{if } w \leq \dfrac{a}{2-k}, \\ 0, & \text{if } w \geq \dfrac{a}{2-k}, \end{cases}$$

$$q_E^r(w) = \begin{cases} \dfrac{a+(1-2k)w}{3b}, & \text{if } w \leq \dfrac{a}{2-k}, \\ \dfrac{a-kw}{2b}, & \text{if } w \geq \dfrac{a}{2-k}, \end{cases}$$

Anticipating the outcome, $U$ sets its $w$ to maximize its profit.

$$\max_w \, (w-c)(q_I(w) + kq_E(w)).$$

First, for $w \leq a/(2-k)$, we derive the "local" optimal input price. The maximization problem is given as

$$\max_w \, (w-c)\left(\frac{a-(2-k)w}{3b} + k\frac{a+(1-2k)w}{3b}\right) \quad s.t. \;\; w \leq \frac{a}{2-k}.$$

This leads to

$$w^r = \frac{(1+k)a + 2(1-k+k^2)c}{4(1-k+k^2)}.$$

This is an interior solution if and only if

$$c \leq \frac{2-5k+5k^2}{2(2-3k+3k^2-k^3)}.$$

The profits of $U$ and $D_I$ are given as

$$\pi_U^r = \frac{(a(1+k) - 2c(k^2-k+1))^2}{24b(k^2-k+1)}, \quad \pi_I^r = \frac{(a(5k^2-5k+2) - 2c(2-k)(k^2-k+1))^2}{144b(k^2-k+1)}.$$

Second, for $w \geq a/(2-k)$, we derive the "local" optimal input price. The maximization problem is given as

$$\max_w \, (w-c)k\frac{a-kw}{2b} \quad s.t. \;\; w \geq \frac{a}{2-k}.$$

This leads to

$$w^r = \frac{a+kc}{2k}.$$

This is an interior solution if and only if

$$c \geq \frac{3k-2}{k(2-k)}.$$

The profits of $U$ and $D_I$ are given as

$$\pi_U^r = \frac{(a-kc)^2}{8b}, \quad \pi_I^r = 0.$$

For any $c \in [0, a)$ such that

$$\frac{3k-2}{k(2-k)} \leq c \leq \frac{2-5k+5k^2}{2(2-3k+3k^2-k^3)},$$

two local optimal input prices exist. We now determine the price that is better for $U$. The first input price leads to higher profits for $U$ if and only if $c < \dot{C}(k)$ (see (2)). Therefore, if $c < \dot{C}(k)$, the optimal input price is the first $w^r$, and then, both $D_I$ and $D_E$ are active. Otherwise, it is the second $w^r$, and then, only $D_E$ is active.

We explore whether an exclusion equilibrium exists by examining whether condition (5) in our paper holds. Substituting the above outcome in the Cournot competition into condition (5) in our paper, we have the condition $0 < c < \bar{C}(k)$ (see (1)) in the result. Therefore, an exclusion equilibrium exists for $0 \leq c < \overline{C}(k)$.

# A3 Adding an inefficient supplier $U_O$

We add an inefficient upstream supplier $U_O$ with a marginal cost $c + c_o$ ($c_o > 0$) to the main model with the linear demand, $Q = (\alpha - p)/\beta$. We assume that

$$\frac{c}{k} < c + c_o < \frac{\alpha}{2},$$

to secure that in each of the upstream and downstream markets, the winner's rival always constrains the winner's pricing.

In Stage 1, $D_I$ offers an exclusive supply contract with fixed compensation of $x_U$ ($x_O$) to $U$ ($U_O$). Moreover, $D_I$ can discriminate the exclusive supply offers between upstream suppliers, and can make the exclusive supply offer to only one of two upstream suppliers. Below, we consider four subgames after upstream suppliers' contractual decisions.

## A3.1 Both $U$ and $U_O$ accept exclusive supply contracts (denote the case $(aa)$)

$D_E$ cannot trade with any supplier. $D_I$ monopolizes the downstream market and solves

$$\max_p \, (p - w)\frac{\alpha - p}{\beta} \quad \Rightarrow \quad p^{aa}(w) = \frac{\alpha + w}{2}.$$

In the upstream market, $U$ wins by setting $w^{aa} = c + c_o$. The profits of the firms are

$$\pi_{DI}^{aa} = \Pi^* (c + c_o) = \frac{(\alpha - c - c_o)^2}{4\beta},$$
$$\pi_{UI}^{aa} = (c + c_o - c) \, Q \, (p^* (c + c_o)) = \frac{c_o (\alpha - c - c_o)}{2\beta}.$$

## A3.2 Neither $U$ nor $U_O$ accepts exclusive supply contracts (denote the case $(rr)$)

$D_E$ enters and wins the downstream competition. The price of $D_E$ is $p^{rr} = w_{DI}$. $U$ wins the upstream competition by setting $w_{DI}^{rr} = w_{DE}^{rr} = c + c_o$. Thus, $p^{rr} = c + c_o$. The profits of the firms are

$$\pi_{DI}^{rr} = 0,$$
$$\pi_U^{rr} = (c + c_o - c) \, kQ \, (c + c_o) = \frac{kc_o (\alpha - c - c_o)}{\beta}, \quad \pi_{UO}^{rr} = 0.$$

### A3.3 Only $U$ accepts an exclusive supply contract (denote the case $(ar)$)

If $D_E$ enters, $U$ and $U_O$ would trade with $D_I$ and $D_E$, respectively. Under the parametric assumption, the pair of $U$ and $D_I$ with marginal cost $c$ would win (the rival's marginal cost is $k(c+c_o)$, which is larger than $c$). Anticipating the result, $D_E$ does not enter.

The outcome in case $ar$ is the same as that in case $aa$. The profits of the firms are

$$\pi_{DI}^{ar} = \Pi^* (c + c_o) = \frac{(\alpha - c - c_o)^2}{4\beta},$$
$$\pi_U^{ar} = (c + c_o - c) Q (p^* (c + c_o)) = \frac{c_o (\alpha - c - c_o)}{2\beta}, \quad \pi_{UO}^{ar} = 0.$$

### A3.4 Only $U_O$ accepts an exclusive supply contract (denote the case $(ra)$)

Anticipating the chance to trade with $U$, $D_E$ enters. In the downstream competition, there are three cases: (i) $c \leq w_U \leq c+c_o$ ($U$ wins the upstream competition and $D_E$ wins the downstream competition by setting $p = w_U$ under the parametric assumption); (ii) $c+c_o < w_U \leq (c+c_o)/k$ ($U$ wins the upstream competition and $D_E$ wins the downstream competition by setting $p = c+c_o$ under the parametric assumption); (iii) $(c+c_o)/k < w_U$ ($U_O$ wins the upstream competition and $D_I$ wins the downstream competition).

We can show that the upper bound of $w_U$ in case (ii) is optimal for $U$. That is, $w_U^{ra} = (c+c_o)/k$. The retail price is $p^{ra} = c + c_o$. The profits of the firms are

$$\pi_{DI}^{ra} = 0,$$
$$\pi_U^{ra} = \left(\frac{c + c_o}{k} - c\right) kQ (c + c_o) = \frac{(c_o + (1-k) c) (\alpha - c - c_o)}{\beta}, \quad \pi_{UO}^{ra} = 0.$$

### A3.5 The condition of exclusive supply contracts

We summarize the above discussions as the following payoff matrix:

| $U \diagdown U_O$ | $a$ | $r$ |
|---|---|---|
| $a$ | $x_U + \frac{c_o (\alpha - c - c_o)}{2\beta}, x_O$ | $x_U + \frac{c_o (\alpha - c - c_o)}{2\beta}, 0$ |
| $r$ | $\frac{(c_o + (1-k) c) (\alpha - c - c_o)}{\beta}, x_O$ | $\frac{kc_o (\alpha - c - c_o)}{\beta}, 0$ |

$U_O$ has an incentive to accept an exclusive supply contract with $x_O$ that is nonnegative. If $D_I$ offers an exclusive supply contract to $U_O$, $D_I$ induces $U_O$ to accept such an exclusive supply contract without any cost. Thus, we omit the discussion on $x_O$.

Furthermore, the following matrix summarizes the relation between $D_E$'s entry decision and the acceptance decisions of $U$ and $U_O$:

| $U \setminus U_O$ | $a$ | $r$ |
|---|---|---|
| $a$ | Exclusion | Exclusion |
| $r$ | Entry | Entry |

#### A3.5.1 The condition that $(a, a)$ is sustainable

We compare the outcome in $(a, a)$ with that in $(r, a)$. We write the conditions of $D_I$ and $U$ to sign an exclusive supply contract:

$$\pi_{DI}^{aa} - x_U \geq \pi_{DI}^{ra} \quad \Leftrightarrow \quad \frac{(\alpha - c - c_o)^2}{4\beta} - x_U \geq 0,$$
$$x_U + \pi_U^{aa} \geq \pi_U^{ra} \quad \Leftrightarrow \quad x_U + \frac{c_o(\alpha - c - c_o)}{2\beta} \geq \frac{(c_o + (1-k)c)(\alpha - c - c_o)}{\beta}.$$

Summing the two inequalities, we obtain:

$$\begin{aligned}\pi_{DI}^{aa} + \pi_U^{aa} - (\pi_{DI}^{ra} + \pi_U^{ra}) &= \frac{(\alpha - c - c_o)^2}{4\beta} + \frac{c_o(\alpha - c - c_o)}{2\beta} - 0 - \frac{(c_o + (1-k)c)(\alpha - c - c_o)}{\beta} \\ &= \frac{(\alpha - c - c_o)(\alpha - 3c_o - (5-4k)c)}{4\beta} \geq 0.\end{aligned}$$

#### A3.5.2 The condition that $(a, r)$ is sustainable

We compare the outcome in $(r, r)$ with that in $(a, r)$.

We write the conditions for $D_I$ and $U_I$ to sign an exclusive supply contract:

$$\pi_{DI}^{ar} - x_U \geq \pi_{DI}^{rr} \quad \Leftrightarrow \quad \frac{(\alpha - c - c_o)^2}{4\beta} - x_U \geq 0$$
$$x_U + \pi_U^{ar} \geq \pi_U^{rr} \quad \Leftrightarrow \quad x_U + \frac{c_o(\alpha - c - c_o)}{2\beta} \geq \frac{kc_o(\alpha - c - c_o)}{\beta}.$$

Summing the two inequalities, we obtain:

$$\begin{aligned}\pi_{DI}^{ar} + \pi_U^{ar} - (\pi_{DI}^{rr} + \pi_U^{rr}) &= \frac{(\alpha - c - c_o)^2}{4\beta} + \frac{c_o(\alpha - c - c_o)}{2\beta} - 0 - \frac{kc_o(\alpha - c - c_o)}{\beta} \\ &= \frac{(\alpha - c - c_o)(\alpha - c + c_o - 4kc_o)}{4\beta} \geq 0.\end{aligned}$$

Solving the inequalities, we obtain:

$$k \leq \frac{\alpha - c + c_o}{4c_o}.$$

### A3.5.3 Optimal contract offer

We compare the lowest fixed payment $x_U$ to achieve $(a, a)$ and $(a, r)$, $x_U^{aa}$ and $x_U^{ar}$, where

$$\begin{aligned} x_U^{aa} &= \pi_U^{ra} - \pi_U^{aa} = \frac{(c_o + (1-k)\,c)\,(\alpha - c - c_o)}{\beta} - \frac{c_o\,(\alpha - c - c_o)}{2\beta}, \\ x_U^{ar} &= \pi_U^{rr} - \pi_U^{ar} = \frac{kc_o\,(\alpha - c - c_o)}{\beta} - \frac{c_o\,(\alpha - c - c_o)}{2\beta}. \end{aligned}$$

The difference between $x_U^{aa}$ and $x_U^{ar}$ is

$$\begin{aligned} x_U^{aa} - x_U^{ar} &= \frac{(c_o + (1-k)\,c)\,(\alpha - c - c_o)}{\beta} - \frac{kc_o\,(\alpha - c - c_o)}{\beta} \\ &= \frac{(1-k)\,(c + c_o)\,(\alpha - c - c_o)}{\beta} > 0. \end{aligned}$$

The *ex ante* profits of $D_I$ in the two cases are the same, $(\alpha - c - c_o)^2/4\beta$. Therefore, $D_I$ prefers $(a, r)$ to $(a, a)$. To achieve $(a, r)$, $D_I$ offers an exclusive supply contract with $x_U^{ar}$ only to $U$. As we have already obtained, the exclusive supply contract is achievable if and only if

$$k \leq \frac{\alpha - c + c_o}{4c_o}.$$

The threshold value is higher than $3/4$ if and only if $\alpha > 2c_o + c$. This inequality always holds because $\alpha > 2(c + c_o)$. Thus, the existence of an inefficient supplier facilitates entry deterrence through exclusive supply contracts.

## A4 A connection between the two polar cases

We consider the linear demand, $Q = \alpha - p$. We assume that when $D_E$ enters the downstream market, $D_I$ can sell its purchased inputs to $D_E$ with a per-unit arbitrage cost $t \geq 0$.

For the case of entry, $U$ indirectly controls the downstream competition by controlling the wholesale price for $D_I$, which is the competitive threat to $D_E$. In the downstream competition, $D_E$ sets its retail price at the wholesale price of $D_I$, $w_I$. The input demand for $U$ is $k(\alpha - w_I)$.

In the upstream market, the maximization problem of $U$ for the case of entry is

$$\max_{w_E, w_I} \ (w_E - c)k(\alpha - w_I), \quad s.t. \ w_I \geq k w_E, \ w_I + t \geq w_E.$$

The first inequality is the condition that $D_E$ becomes the winner; the second inequality is the condition that there is no arbitrage between $D_I$ and $D_E$.

The Lagrange function is

$$\mathcal{L} = (w_E - c)k(\alpha - w_I) + \lambda_I(w_I - k w_E) + \lambda_A(w_I + t - w_E).$$

We have the three situations with respect to $t$.

**Situation (i): low arbitrage cost** When $t \leq (1-k)(\alpha+c)/(1+k)$, the solution is

$$w_E^{r(i)} = \frac{\alpha + c + t}{2}, \ w_I^{r(i)} = \frac{\alpha + c - t}{2}, \ \lambda_I^{r(i)} = 0, \ \lambda_A^{r(i)} = \frac{k(\alpha - c + t)}{2}.$$

The joint profit of $D_I$ and $U$ for the case of entry becomes

$$\pi_I^{r(i)} + \pi_U^{r(i)} = \frac{k(\alpha - c + t)^2}{4}.$$

Note that under exclusive dealing, the joint profit of $D_I$ and $U$ becomes

$$\pi_I^a + \pi_U^a = \frac{3(\alpha - c)^2}{16}.$$

Using the above equations, condition (5) in our paper holds if and only if

$$t < \min\left\{ \frac{\left(\sqrt{3k} - 2k\right)(\alpha - c)}{2k}, \frac{(1-k)(\alpha+c)}{1+k} \right\} \ \ and \ \ k \leq \frac{3}{4}.$$

Thus, when input arbitrage is costless, exclusion can be achievable for lower $k$.

**Situation (ii): intermediate arbitrage cost** When $(1-k)(\alpha+c)/(1+k) \leq t \leq (1-k)(\alpha+kc)/(2k)$, the solution is

$$\begin{aligned} &w_E^{r(ii)} = \frac{t}{1-k}, \ w_I^{r(ii)} = \frac{kt}{1-k}, \\ &\lambda_I^{r(ii)} = \frac{k((1+k)t - (1-k)(\alpha+c))}{(1-k)^2}, \ \lambda_A^{r(ii)} = \frac{k((1-k)(\alpha+kc) - 2kt)}{(1-k)^2}. \end{aligned}$$

The joint profit of $D_I$ and $U$ for the case of entry becomes

$$\pi_I^{r(ii)} + \pi_U^{r(ii)} = \frac{k((1-k)\alpha - kt)(t-(1-k)c)}{(1-k)^2}.$$

Condition (5) in our paper holds if and only if

$$t < \frac{(1-k)\left(2(\alpha+kc) - \sqrt{\alpha^2 + 2(3-4k)\alpha c - (3-4k^2)c^2}\right)}{4k}.$$

The threshold value of $t$ is always smaller than the upper bound of this case. Furthermore, the threshold value of $t$ is larger than the lower bound of this case if and only if

$$c < \frac{(3+6k-13k^2-4(1-k^2)\sqrt{3k})\alpha}{3+6k+3k^2-16k^3} \quad and \quad k < \frac{1}{3}.$$

Thus, we can obtain exclusion outcomes for lower values of $c$ and $k$.

**Situation (iii): high arbitrage cost** When $(1-k)(\alpha+kc)/(2k) \leq t$, the solution is

$$w_E^{r(iii)} = \frac{\alpha+kc}{2k}, \; w_I^{r(iii)} = \frac{\alpha+kc}{2}, \; \lambda_I^{r(iii)} = \frac{\alpha-kc}{2}, \; \lambda_A^{r(iii)} = 0.$$

The joint profit of $D_I$ and $U$ for the case of entry becomes

$$\pi_I^{r(iii)} + \pi_U^{r(iii)} = \frac{(\alpha-kc)^2}{4},$$

which coincides with $U$'s profit under perfect input price discrimination, provided in Section 5.2 in our paper. Thus, exclusion is not achievable.

# A5 Differentiated products

We explain the mathematical procedure to derive the result in the second paragraph of Section 4.4 in the main text. From the next page, the detailed analysis made with Mathematica is available.

# Differentiated products

We consider two cases: (i) DI excludes DE; (ii) DE enters the market.
Finally, we compare the outcomes in the two cases.

## (i) The case in which DI excludes DE.

Solving the first-order condition of DI's maximization problem, we have

```
Factor[D[(pi - w) (a - pi), pi]]
```

$a - 2\,pi + w$

```
Factor[Solve[a - 2 pi + w == 0, pi]]
```

$\left\{\left\{pi \to \frac{a+w}{2}\right\}\right\}$

Solving the first-order condition of U's maximization problem, we have

$$\texttt{Factor}\left[\texttt{D}\left[(w-c)\,(a-pi)\ /.\ \left\{pi \to \frac{a+w}{2}\right\},\ w\right]\right]$$

$\frac{1}{2}\,(a + c - 2\,w)$

$$\texttt{Factor}\left[\texttt{Solve}\left[\frac{1}{2}\,(a+c-2\,w) == 0,\ w\right]\right]$$

$\left\{\left\{w \to \frac{a+c}{2}\right\}\right\}$

We obtain the profits of D1 and U respectively:

$$\texttt{Factor}\left[(pi-w)\,(a-pi)\ /.\ \left\{pi \to \frac{a+w}{2}\right\}\ /.\ \left\{w \to \frac{a+c}{2}\right\}\right]$$
$$\texttt{Factor}\left[(w-c)\,(a-pi)\ /.\ \left\{pi \to \frac{a+w}{2}\right\}\ /.\ \left\{w \to \frac{a+c}{2}\right\}\right]$$

$\frac{1}{16}\,(a-c)^2$

$\frac{1}{8}\,(a-c)^2$

## (ii) The case in which DE enters.

The demand system in this setting

```
Solve[{pi == a - qi - γ qe, pe == a - qe - γ qi}, {qe, qi}]
```

$\left\{\left\{qe \to -\frac{a - pe - a\,\gamma + pi\,\gamma}{-1+\gamma^2},\ qi \to -\frac{a - pi - a\,\gamma + pe\,\gamma}{-1+\gamma^2}\right\}\right\}$

Given the common wholesale price, the first-order conditions of the downstream firms are:

$$\text{Factor}\left[D\left[(\text{pe}-k\,w)\left(\frac{a-\text{pe}-a\gamma+\text{pi}\,\gamma}{1-\gamma^2}\right),\ \text{pe}\right]\right]$$

$$\text{Factor}\left[D\left[(\text{pi}-w)\left(\frac{a-\text{pi}-a\gamma+\text{pe}\,\gamma}{1-\gamma^2}\right),\ \text{pi}\right]\right]$$

$$\frac{-a+2\,\text{pe}-k\,w+a\gamma-\text{pi}\,\gamma}{(-1+\gamma)\,(1+\gamma)}$$

$$\frac{-a+2\,\text{pi}-w+a\gamma-\text{pe}\,\gamma}{(-1+\gamma)\,(1+\gamma)}$$

Solving the first-order conditions, we have the candidate of the prices.

$$\text{Factor}\left[\text{Solve}\left[\left\{\frac{-a+2\,\text{pe}-k\,w+a\gamma-\text{pi}\,\gamma}{(-1+\gamma)\,(1+\gamma)}==0,\ \frac{-a+2\,\text{pi}-w+a\gamma-\text{pe}\,\gamma}{(-1+\gamma)\,(1+\gamma)}==0\right\},\ \{\text{pi},\text{pe}\}\right]\right]$$

$$\left\{\left\{\text{pi}\to\frac{-2a-2w+a\gamma-k\,w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)},\ \text{pe}\to\frac{-2a-2k\,w+a\gamma-w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)}\right\}\right\}$$

Substituting them into the demand functions, we derive the quantities:

$$\text{Factor}\left[\left\{\text{qe}\to-\frac{a-\text{pe}-a\gamma+\text{pi}\,\gamma}{-1+\gamma^2},\ \text{qi}\to-\frac{a-\text{pi}-a\gamma+\text{pe}\,\gamma}{-1+\gamma^2}\right\}\ /.\ \left\{\text{pi}\to\frac{-2a-2w+a\gamma-k\,w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)},\ \text{pe}\to\frac{-2a-2k\,w+a\gamma-w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)}\right\}\right]$$

$$\left\{\text{qe}\to-\frac{-2a+2k\,w+a\gamma-w\gamma+a\gamma^2-k\,w\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)},\ \text{qi}\to-\frac{-2a+2w+a\gamma-k\,w\gamma+a\gamma^2-w\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)}\right\}$$

We check the condition that the outcome is an interior solution:

$$\text{Factor}\left[\text{Solve}\left[-\frac{-2a+2w+a\gamma-k\,w\gamma+a\gamma^2-w\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)}==0,\ w\right]\right]$$

$$\left\{\left\{w\to\frac{a\,(-1+\gamma)\,(2+\gamma)}{-2+k\gamma+\gamma^2}\right\}\right\}$$

The outcome is applicable if and only if $w < \frac{a\,(1-\gamma)\,(2+\gamma)}{2-k\,\gamma-\gamma^2}$.

Under the interior solution, the profits of the firms are

$$\text{Factor}\left[(\text{pe}-k\,w)\left(\frac{a-\text{pe}-a\gamma+\text{pi}\,\gamma}{1-\gamma^2}\right)\ /.\ \left\{\text{pi}\to\frac{-2a-2w+a\gamma-k\,w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)},\ \text{pe}\to\frac{-2a-2k\,w+a\gamma-w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)}\right\}\right]$$

$$\text{Factor}\left[(\text{pi}-w)\left(\frac{a-\text{pi}-a\gamma+\text{pe}\,\gamma}{1-\gamma^2}\right)\ /.\ \left\{\text{pi}\to\frac{-2a-2w+a\gamma-k\,w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)},\ \text{pe}\to\frac{-2a-2k\,w+a\gamma-w\gamma+a\gamma^2}{(-2+\gamma)\,(2+\gamma)}\right\}\right]$$

$$-\frac{\left(2a-2k\,w-a\gamma+w\gamma-a\gamma^2+k\,w\gamma^2\right)^2}{(-2+\gamma)^2\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)^2}$$

$$-\frac{\left(-2a+2w+a\gamma-k\,w\gamma+a\gamma^2-w\gamma^2\right)^2}{(-2+\gamma)^2\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)^2}$$

We consider the case in which $w > \frac{a\,(1-\gamma)\,(2+\gamma)}{2-k\,\gamma-\gamma^2}$ (DE monopoly).

The first-order condition of DE under the constraint in which DI is out of the market:

$$\text{Factor}\left[D\left[(pe-k\,w)\,(a-pe)-\lambda\,\frac{a-pi-a\,\gamma+pe\,\gamma}{1-\gamma^2}\;/.\;pi\to w,\;pe\right]\right]$$

$$\frac{-a+2\,pe-k\,w+a\,\gamma^2-2\,pe\,\gamma^2+k\,w\,\gamma^2+\gamma\,\lambda}{(-1+\gamma)\,(1+\gamma)}$$

Solving the first-order condition, we have the candidate of the price

$$\text{Factor}[\text{Solve}[\%==0,\;pe]]$$

$$\left\{\left\{pe\to\frac{-a-k\,w+a\,\gamma^2+k\,w\,\gamma^2+\gamma\,\lambda}{2\,(-1+\gamma)\,(1+\gamma)}\right\}\right\}$$

We check the condition that the constraint is binding

$$\text{Factor}\left[\text{Solve}\left[\frac{a-pi-a\,\gamma+pe\,\gamma}{1-\gamma^2}==0\;/.\;pi\to w\;/.\;pe\to\frac{-a-k\,w+a\,\gamma^2+k\,w\,\gamma^2+\gamma\,\lambda}{2\,(-1+\gamma)\,(1+\gamma)},\;\lambda\right]\right]$$

$$\left\{\left\{\lambda\to\frac{(-1+\gamma)\,(1+\gamma)\,(-2\,a+2\,w+a\,\gamma-k\,w\,\gamma)}{\gamma^2}\right\}\right\}$$

We check when the multiplier is positive:

$$\text{Solve}\left[\frac{(-1+\gamma)\,(1+\gamma)\,(-2\,a+2\,w+a\,\gamma-k\,w\,\gamma)}{\gamma^2}==0,\;w\right]$$

$$\left\{\left\{w\to\frac{a\,(-2+\gamma)}{-2+k\,\gamma}\right\}\right\}$$

We have the prices under which the constraint is binding / non-binding.

$$\text{Factor}\left[qe\to(a-pe)\;/.\;pe\to\frac{-a-k\,w+a\,\gamma^2+k\,w\,\gamma^2+\gamma\,\lambda}{2\,(-1+\gamma)\,(1+\gamma)}\;/.\;\lambda\to\frac{(-1+\gamma)\,(1+\gamma)\,(-2\,a+2\,w+a\,\gamma-k\,w\,\gamma)}{\gamma^2}\right]$$

$$\text{Factor}\left[qe\to(a-pe)\;/.\;pe\to\frac{-a-k\,w+a\,\gamma^2+k\,w\,\gamma^2+\gamma\,\lambda}{2\,(-1+\gamma)\,(1+\gamma)}\;/.\;\lambda\to 0\right]$$

$$qe\to\frac{a-w}{\gamma}$$

$$qe\to\frac{1}{2}\,(a-k\,w)$$

The price of DE is the former if $\frac{a\,(1-\gamma)\,(2+\gamma)}{2-k\,\gamma-\gamma^2} < w < \frac{a\,(2-\gamma)}{2-k\,\gamma}$, the price of DE is the latter if $\frac{a\,(2-\gamma)}{2-k\,\gamma} \le w$. In other words, if $\frac{a\,(1-\gamma)\,(2+\gamma)}{2-k\,\gamma-\gamma^2} < w < \frac{a\,(2-\gamma)}{2-k\,\gamma}$, DI works as a constraint; if $\frac{a\,(2-\gamma)}{2-k\,\gamma} \le w$, DE sets the unconstrained monopoly price.

The profits of DE under the former and latter prices are

$$\text{Factor}\left[(\text{pe} - \text{k w})\ (\text{a} - \text{pe}) - \lambda\ \frac{\text{a} - \text{pi} - \text{a}\gamma + \text{pe}\gamma}{1 - \gamma^2}\ /.\ \text{pi} \to \text{w}\ /.\ \text{pe} \to \frac{-\text{a} - \text{k w} + \text{a}\gamma^2 + \text{k w}\gamma^2 + \gamma\lambda}{2\ (-1 + \gamma)\ (1 + \gamma)}\ /.\ \lambda \to \frac{(-1 + \gamma)\ (1 + \gamma)\ (-2\text{a} + 2\text{w} + \text{a}\gamma - \text{k w}\gamma)}{\gamma^2}\right]$$

$$\text{Factor}\left[(\text{pe} - \text{k w})\ (\text{a} - \text{pe}) - \lambda\ \frac{\text{a} - \text{pi} - \text{a}\gamma + \text{pe}\gamma}{1 - \gamma^2}\ /.\ \text{pi} \to \text{w}\ /.\ \text{pe} \to \frac{-\text{a} - \text{k w} + \text{a}\gamma^2 + \text{k w}\gamma^2 + \gamma\lambda}{2\ (-1 + \gamma)\ (1 + \gamma)}\ /.\ \lambda \to 0\right]$$

$$\frac{(\text{a} - \text{w})\ (-\text{a} + \text{w} + \text{a}\gamma - \text{k w}\gamma)}{\gamma^2}$$

$$\frac{1}{4}\ (\text{a} - \text{k w})^2$$

We check the difference between the profits under the duopoly and the constrained monopoly for the threshold wholesale price $\text{w} \to \frac{\text{a}\ (1-\gamma)\ (2+\gamma)}{2-\text{k}\gamma-\gamma^2}$:

$$\text{Factor}\left[-\frac{(2\text{a} - 2\text{k w} - \text{a}\gamma + \text{w}\gamma - \text{a}\gamma^2 + \text{k w}\gamma^2)^2}{(-2 + \gamma)^2\ (-1 + \gamma)\ (1 + \gamma)\ (2 + \gamma)^2} - \frac{(\text{a} - \text{w})\ (-\text{a} + \text{w} + \text{a}\gamma - \text{k w}\gamma)}{\gamma^2}\ /.\ \text{w} \to \frac{\text{a}\ (1 - \gamma)\ (2 + \gamma)}{2 - \text{k}\gamma - \gamma^2}\right]$$

$$0$$

From the discussion, we have three subgames:
(**Case 1**) Duopoly $\text{w} < \frac{\text{a}\ (1-\gamma)\ (2+\gamma)}{2-\text{k}\gamma-\gamma^2}$,
(**Case 2**) Constrained monopoly by DE $\frac{\text{a}\ (1-\gamma)\ (2+\gamma)}{2-\text{k}\gamma-\gamma^2} \le \text{w} < \frac{\text{a}\ (2-\gamma)}{2-\text{k}\gamma}$,
(**Case 3**) Unconstrained monopoly by DE $\frac{\text{a}\ (2-\gamma)}{2-\text{k}\gamma} \le \text{w}$.

For analytical simplicity, we assume that c=0.

```
c = 0
```

```
0
```

### Case 1

The first-order condition of U under the constraint $\text{w} < \frac{\text{a}\ (1-\gamma)\ (2+\gamma)}{2-\text{k}\gamma-\gamma^2}$ is

$$\text{Factor}\left[\text{D}\left[(\text{w} - \text{c})\ (\text{qi} + \text{k qe}) + \lambda\left(\frac{\text{a}\ (-1 + \gamma)\ (2 + \gamma)}{-2 + \text{k}\gamma + \gamma^2} - \text{w}\right)\ /.\ \left\{\text{qe} \to -\frac{-2\text{a} + 2\text{k w} + \text{a}\gamma - \text{w}\gamma + \text{a}\gamma^2 - \text{k w}\gamma^2}{(-2 + \gamma)\ (-1 + \gamma)\ (1 + \gamma)\ (2 + \gamma)},\ \text{qi} \to -\frac{-2\text{a} + 2\text{w} + \text{a}\gamma - \text{k w}\gamma + \text{a}\gamma^2 - \text{w}\gamma^2}{(-2 + \gamma)\ (-1 + \gamma)\ (1 + \gamma)\ (2 + \gamma)}\right\},\ \text{w}\right]\right]$$

$$-\frac{-2\text{a} - 2\text{a k} + 4\text{w} + 4\text{k}^2\text{w} + \text{a}\gamma + \text{a k}\gamma - 4\text{k w}\gamma + \text{a}\gamma^2 + \text{a k}\gamma^2 - 2\text{w}\gamma^2 - 2\text{k}^2\text{w}\gamma^2 + 4\lambda - 5\gamma^2\lambda + \gamma^4\lambda}{(-2 + \gamma)\ (-1 + \gamma)\ (1 + \gamma)\ (2 + \gamma)}$$

Solving it, we have the candidate of the wholesale price:

```
Factor[Solve[% == 0, w]]
```

$$\left\{\left\{\text{w} \to \frac{(-1 + \gamma)\ (2 + \gamma)\ (\text{a} + \text{a k} - 2\lambda - \gamma\lambda + \gamma^2\lambda)}{2\ (-2 - 2\text{k}^2 + 2\text{k}\gamma + \gamma^2 + \text{k}^2\gamma^2)}\right\}\right\}$$

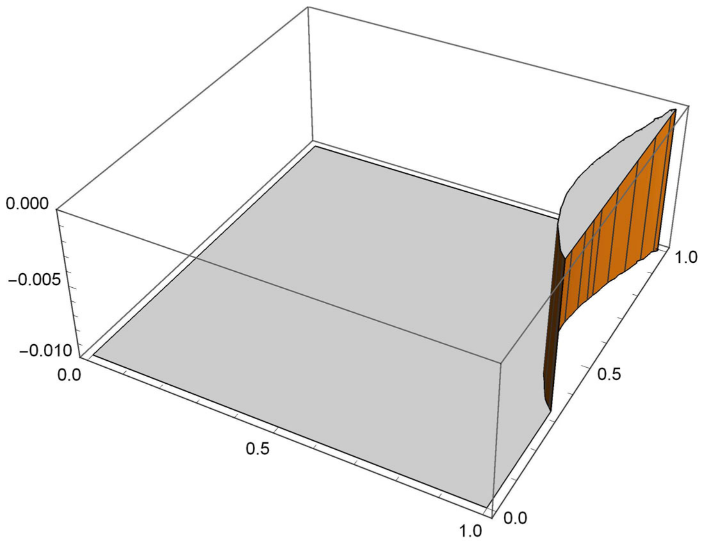
We obtain the multiplier

```
Simplify[Solve[a (-1 + γ) (2 + γ) / (-2 + k γ + γ^2) - w == 0 /. %, λ]]
```

$$\left\{\left\{\lambda \to \frac{a\left(-2+\gamma^2+k\left(2+3\gamma-\gamma^2\right)+k^2\left(-4-\gamma+2\gamma^2\right)\right)}{\left(-2-\gamma+\gamma^2\right)\left(-2+k\gamma+\gamma^2\right)}\right\}\right\}$$

We graphically show the condition that the multiplier is larger than zero.

```
Plot3D[(-2 + γ^2 + k (2 + 3 γ - γ^2) + k^2 (-4 - γ + 2 γ^2)) / ((-2 - γ + γ^2) (-2 + k γ + γ^2)),
 {γ, 0, 1}, {k, 0, 1}, PlotRange → {-0.01, 0}]
```




We explicitly derive the condition that the multiplier is larger than zero.

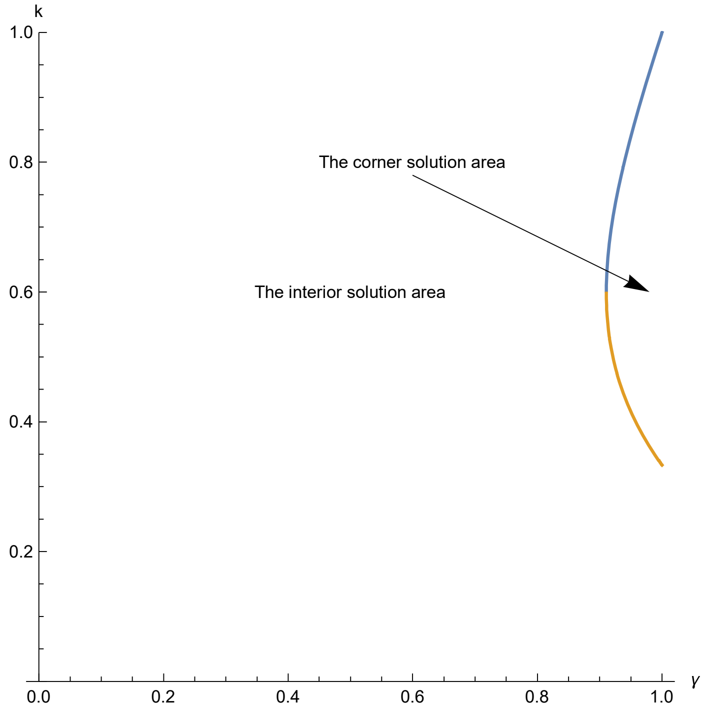
```
Factor[Solve[(-2 + γ^2 + k (2 + 3 γ - γ^2) + k^2 (-4 - γ + 2 γ^2)) / ((-2 - γ + γ^2) (-2 + k γ + γ^2)) == 0, k]]
```

$$\left\{\left\{k \to -\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)}\right\},\right.$$
$$\left.\left\{k \to \frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)}\right\}\right\}$$

The following figure summarizes the condition that the solution is interior (corner).

$$\text{Plot}\left[\left\{-\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\right.\right.$$
$$\left.\frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)}\right\}, \{\gamma, 0, 1\}, \text{PlotRange} \to \{0, 1\},$$

```
AxesLabel → {"γ", "k"}, Epilog → {Text["The interior solution area", {0.5, 0.6}],
  Text["The corner solution area", {0.6, 0.8}], Arrow[{{0.6, 0.78}, {0.98, 0.6}}]},
AxesOrigin → {0, 0}, AspectRatio → 1]
```




In the interior solution, the profits of U and DI, the downstream quantities, and the wholesale price are

$$\text{Factor}\Big[(w-c)\,(qi+k\,qe)+\lambda\left(\frac{a\,(-1+\gamma)\,(2+\gamma)}{-2+k\,\gamma+\gamma^2}-w\right)\,/.\,\Big\{qe\to-\frac{-2\,a+2\,k\,w+a\,\gamma-w\,\gamma+a\,\gamma^2-k\,w\,\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)},$$

$$qi\to-\frac{-2\,a+2\,w+a\,\gamma-k\,w\,\gamma+a\,\gamma^2-w\,\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)}\Big\}\,/.\,\Big\{w\to$$

$$\frac{-2\,a-2\,c-2\,a\,k-2\,c\,k^2+a\,\gamma+a\,k\,\gamma+2\,c\,k\,\gamma+a\,\gamma^2+c\,\gamma^2+a\,k\,\gamma^2+c\,k^2\,\gamma^2+4\,\lambda-5\,\gamma^2\,\lambda+\gamma^4\,\lambda}{2\,\left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)}\Big\}$$

$$/.\;\lambda\to 0\Big]$$

$$\text{Factor}\Big[(pi-w)\left(\frac{a-pi-a\,\gamma+pe\,\gamma}{1-\gamma^2}\right)\,/.\,\Big\{pi\to\frac{-2\,a-2\,w+a\,\gamma-k\,w\,\gamma+a\,\gamma^2}{(-2+\gamma)\,(2+\gamma)},$$

$$pe\to\frac{-2\,a-2\,k\,w+a\,\gamma-w\,\gamma+a\,\gamma^2}{(-2+\gamma)\,(2+\gamma)}\Big\}\,/.\,\Big\{w\to$$

$$\frac{-2\,a-2\,c-2\,a\,k-2\,c\,k^2+a\,\gamma+a\,k\,\gamma+2\,c\,k\,\gamma+a\,\gamma^2+c\,\gamma^2+a\,k\,\gamma^2+c\,k^2\,\gamma^2+4\,\lambda-5\,\gamma^2\,\lambda+\gamma^4\,\lambda}{2\,\left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)}\Big\}$$

$$/.\;\lambda\to 0\Big]$$

$$\text{Simplify}\Big[\Big\{qe\to-\frac{-2\,a+2\,k\,w+a\,\gamma-w\,\gamma+a\,\gamma^2-k\,w\,\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)},$$

$$qi\to-\frac{-2\,a+2\,w+a\,\gamma-k\,w\,\gamma+a\,\gamma^2-w\,\gamma^2}{(-2+\gamma)\,(-1+\gamma)\,(1+\gamma)\,(2+\gamma)}\Big\}\,/.\,\Big\{w\to$$

$$\frac{-2\,a-2\,c-2\,a\,k-2\,c\,k^2+a\,\gamma+a\,k\,\gamma+2\,c\,k\,\gamma+a\,\gamma^2+c\,\gamma^2+a\,k\,\gamma^2+c\,k^2\,\gamma^2+4\,\lambda-5\,\gamma^2\,\lambda+\gamma^4\,\lambda}{2\,\left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)}\Big\}$$

$$/.\;\lambda\to 0\Big]$$

$$\text{Simplify}\Big[$$

$$\Big\{w\to\frac{-2\,a-2\,c-2\,a\,k-2\,c\,k^2+a\,\gamma+a\,k\,\gamma+2\,c\,k\,\gamma+a\,\gamma^2+c\,\gamma^2+a\,k\,\gamma^2+c\,k^2\,\gamma^2+4\,\lambda-5\,\gamma^2\,\lambda+\gamma^4\,\lambda}{2\,\left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)}\Big\}$$

$$/.\;\lambda\to 0\Big]$$

$$-\frac{a^2\,(1+k)^2\,(-1+\gamma)\,(2+\gamma)}{4\,(-2+\gamma)\,(1+\gamma)\,\left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)}$$

$$-\frac{a^2\,(-1+\gamma)\,\left(-2+2\,k-4\,k^2+3\,k\,\gamma-k^2\,\gamma+\gamma^2-k\,\gamma^2+2\,k^2\,\gamma^2\right)^2}{4\,(-2+\gamma)^2\,(1+\gamma)\,\left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)^2}$$

$$\left\{qe \to \frac{a\left(4+\gamma-2\gamma^2-k^2\left(-2+\gamma^2\right)+k\left(-2-3\gamma+\gamma^2\right)\right)}{2\,(-2+\gamma)\,(1+\gamma)\left(-2+2k\gamma+\gamma^2+k^2\left(-2+\gamma^2\right)\right)},\right.$$

$$\left.qi \to \frac{a\left(2-\gamma^2+k^2\left(4+\gamma-2\gamma^2\right)+k\left(-2-3\gamma+\gamma^2\right)\right)}{2\,(-2+\gamma)\,(1+\gamma)\left(-2+2k\gamma+\gamma^2+k^2\left(-2+\gamma^2\right)\right)}\right\}$$

$$\left\{w \to \frac{a\,(1+k)\left(-2+\gamma+\gamma^2\right)}{2\left(-2+2k\gamma+\gamma^2+k^2\left(-2+\gamma^2\right)\right)}\right\}$$

In the corner solution, the profit of U is (the profit of DI is zero), the downstream quantities, and the wholesale price are:

```
Factor[
 (w - c) (qi + k qe) + λ (a (-1 + γ) (2 + γ)/(-2 + k γ + γ^2) - w) /. {qe -> -(-2 a + 2 k w + a γ - w γ + a γ^2 - k w γ^2)/((-2 + γ) (-1 + γ) (1 + γ) (2 + γ)),
    qi -> -(-2 a + 2 w + a γ - k w γ + a γ^2 - w γ^2)/((-2 + γ) (-1 + γ) (1 + γ) (2 + γ))} /. {w ->
     (-2 a - 2 c - 2 a k - 2 c k^2 + a γ + a k γ + 2 c k γ + a γ^2 + c γ^2 + a k γ^2 + c k^2 γ^2 + 4 λ - 5 γ^2 λ + γ^4 λ)/(2 (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2))}
   /. {λ -> (a (-2 + γ + γ^2) (-2 + γ^2 + k (2 + 3 γ - γ^2) + k^2 (-4 - γ + 2 γ^2)) -
       c (3 k γ (-2 + γ^2) + k^3 γ (-2 + γ^2) + (-2 + γ^2)^2 + k^2 (4 - 2 γ^2 + γ^4)))/
     ((-2 + k γ + γ^2) (4 - 5 γ^2 + γ^4))}]
Factor[0]
Simplify[
 {qe -> -(-2 a + 2 k w + a γ - w γ + a γ^2 - k w γ^2)/((-2 + γ) (-1 + γ) (1 + γ) (2 + γ)), qi -> -(-2 a + 2 w + a γ - k w γ + a γ^2 - w γ^2)/((-2 + γ) (-1 + γ) (1 + γ) (2 + γ))} /. {w ->
     (-2 a - 2 c - 2 a k - 2 c k^2 + a γ + a k γ + 2 c k γ + a γ^2 + c γ^2 + a k γ^2 + c k^2 γ^2 + 4 λ - 5 γ^2 λ + γ^4 λ)/(2 (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2))}
   /. {λ -> (a (-2 + γ + γ^2) (-2 + γ^2 + k (2 + 3 γ - γ^2) + k^2 (-4 - γ + 2 γ^2)) -
       c (3 k γ (-2 + γ^2) + k^3 γ (-2 + γ^2) + (-2 + γ^2)^2 + k^2 (4 - 2 γ^2 + γ^4)))/
     ((-2 + k γ + γ^2) (4 - 5 γ^2 + γ^4))}]
Simplify[
 {w ->
    (-2 a - 2 c - 2 a k - 2 c k^2 + a γ + a k γ + 2 c k γ + a γ^2 + c γ^2 + a k γ^2 + c k^2 γ^2 + 4 λ - 5 γ^2 λ + γ^4 λ)/(2 (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2))}
  /. {λ -> (a (-2 + γ + γ^2) (-2 + γ^2 + k (2 + 3 γ - γ^2) + k^2 (-4 - γ + 2 γ^2)) -
      c (3 k γ (-2 + γ^2) + k^3 γ (-2 + γ^2) + (-2 + γ^2)^2 + k^2 (4 - 2 γ^2 + γ^4)))/
    ((-2 + k γ + γ^2) (4 - 5 γ^2 + γ^4))}]
```

$$\frac{a^2 (-1+k)\, k\, (-1+\gamma)\, (2+\gamma)}{\left(-2+k\gamma+\gamma^2\right)^2}$$

$$\left\{qe \to \frac{a\,(-1+k)}{-2+k\gamma+\gamma^2},\ qi \to 0\right\}$$

$$\left\{w \to \frac{a\left(-2+\gamma+\gamma^2\right)}{-2+k\gamma+\gamma^2}\right\}$$

## Case 2

The first-order condition of U under the constraint $\frac{a\,(1-\gamma)\,(2+\gamma)}{2-k\,\gamma-\gamma^2} < w < \frac{a\,(2-\gamma)}{2-k\,\gamma}$ is

$$\text{Factor}\left[D\left[(w-c)\ (qi + k\,qe) + \lambda a\left(w - \frac{a\ (-1+\gamma)\ (2+\gamma)}{-2+k\,\gamma+\gamma^2}\right) + \lambda b\left(\frac{a\ (-2+\gamma)}{-2+k\,\gamma} - w\right) /.\ \left\{qe \to \frac{a-w}{\gamma},\ qi \to 0\right\},\ w\right]\right]$$

$$\frac{a\,k - 2\,k\,w + \gamma\,\lambda a - \gamma\,\lambda b}{\gamma}$$

Solving it, we have the candidate of the wholesale price:

$$\text{Solve}\left[\frac{a\,k + c\,k - 2\,k\,w + \gamma\,\lambda a - \gamma\,\lambda b}{\gamma} == 0,\ w\right]$$

$$\left\{\left\{w \to \frac{a\,k + c\,k + \gamma\,\lambda a - \gamma\,\lambda b}{2\,k}\right\}\right\}$$

We obtain the multipliers

$$\text{Factor}\left[\text{Solve}\left[\left\{w - \frac{a\ (-1+\gamma)\ (2+\gamma)}{-2+k\,\gamma+\gamma^2} == 0,\ \lambda b == 0\right\} /.\ \left\{w \to \frac{a\,k + c\,k + \gamma\,\lambda a - \gamma\,\lambda b}{2\,k}\right\},\ \{\lambda a,\ \lambda b\}\right]\right]$$

$$\text{Factor}\left[\text{Solve}\left[\left\{\frac{a\ (-2+\gamma)}{-2+k\,\gamma} - w == 0,\ \lambda a == 0\right\} /.\ \left\{w \to \frac{a\,k + c\,k + \gamma\,\lambda a - \gamma\,\lambda b}{2\,k}\right\},\ \{\lambda a,\ \lambda b\}\right]\right]$$

$$\left\{\left\{\lambda a \to -\frac{a\,k\ \left(2 - 2\,\gamma + k\,\gamma - \gamma^2\right)}{\gamma\ \left(-2 + k\,\gamma + \gamma^2\right)},\ \lambda b \to 0\right\}\right\}$$

$$\left\{\left\{\lambda a \to 0,\ \lambda b \to \frac{a\,k\ (2 - 2\,\gamma + k\,\gamma)}{\gamma\ (-2 + k\,\gamma)}\right\}\right\}$$

Note that $\lambda$b is always negative.
We check the sign of $\lambda$a:

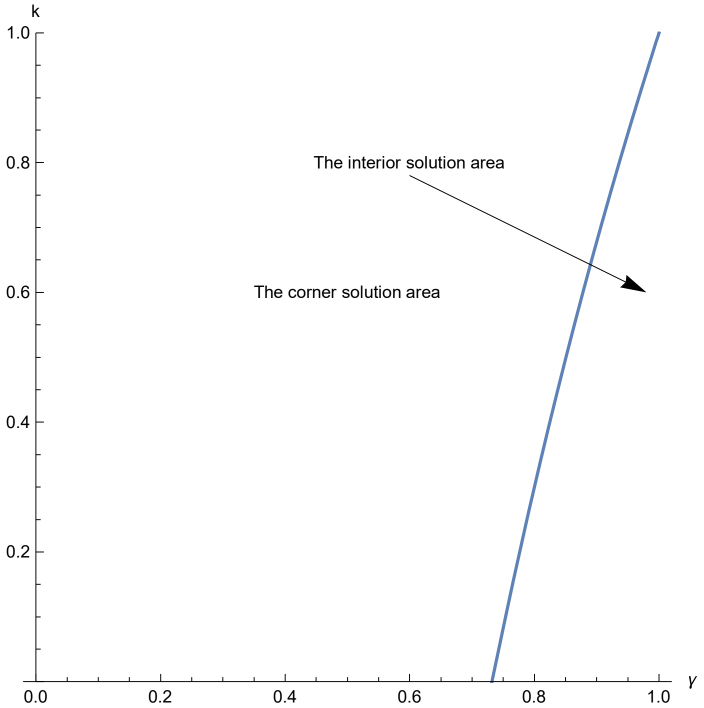
$$\text{Simplify}\left[\text{Solve}\left[-\frac{a\,k\ \left(2 - 2\,\gamma + k\,\gamma - \gamma^2\right)}{\gamma\ \left(-2 + k\,\gamma + \gamma^2\right)} == 0,\ k\right]\right]$$

$$\left\{\{k \to 0\},\ \left\{k \to 2 - \frac{2}{\gamma} + \gamma\right\}\right\}$$

This case has an interior solution if and only if $k < 2 - \frac{2}{\gamma} + \gamma$

The following figure summarizes the condition that the solution is interior (corner)

```
Plot[2 - 2/γ + γ, {γ, 0, 1}, PlotRange → {0, 1}, AxesLabel → {"γ", "k"},
 Epilog → {Text["The corner solution area", {0.5, 0.6}],
   Text["The interior solution area", {0.6, 0.8}], Arrow[{{0.6, 0.78}, {0.98, 0.6}}]},
 AxesOrigin → {0, 0}, AspectRatio → 1]
```



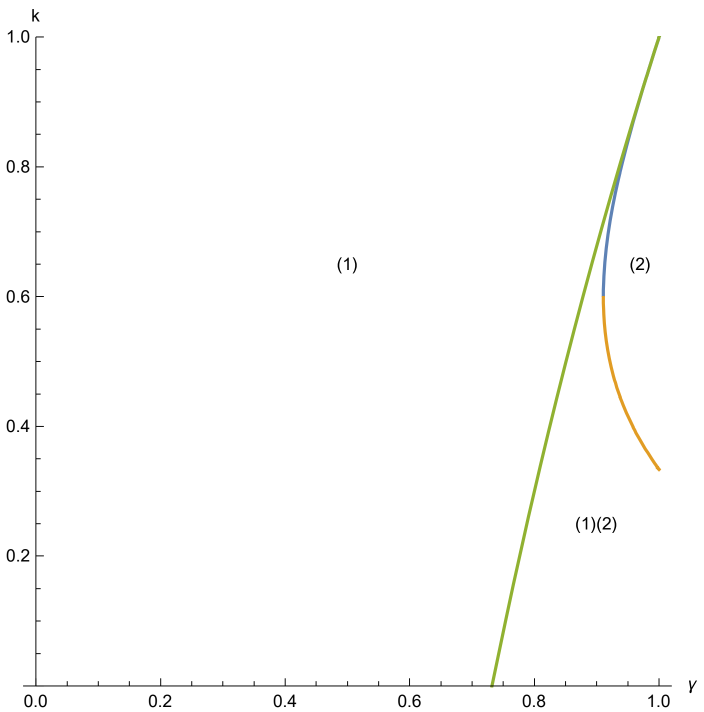

We merge the two figures to express the conditions that the solutions are interior (corner) in Cases 1 and 2:
(1) and (2) indicate that the areas are interior in case 1 and 2 respectively.

$$\texttt{Plot}\left[\left\{-\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\ \frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\ 2-\frac{2}{\gamma}+\gamma\right\},\right.$$

```
{γ, 0, 1}, PlotRange → {0, 1}, AxesLabel → {"γ", "k"}, Epilog →
  {Text["(1)", {0.5, 0.65}], Text["(1)(2)", {0.9, 0.25}], Text["(2)", {0.97, 0.65}]},
 AxesOrigin → {0, 0}, AspectRatio → 1]
```

$$\texttt{Plot}\left[\left\{2-\frac{2}{\gamma}+\gamma-\left(-\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)}\right)\right\},\right.$$

```
{γ, 0.95, 1}, PlotRange → {0, 0.001}]
```




### Case 3

The first-order condition of U is

$$\texttt{Factor}\left[\texttt{D}\left[(w-c)\ (qi+k\,qe)+\lambda\left(w-\frac{a\ (-2+\gamma)}{-2+k\gamma}\right)\ /.\ \left\{qe\to\frac{1}{2}\ (a-k\,w),\ qi\to 0\right\},\ w\right]\right]$$

$$\frac{1}{2}\left(a\,k-2\,k^2\,w+2\,\lambda\right)$$

Solving it, we have the candidate of the wholesale price:

$$\texttt{Solve}\left[\frac{1}{2}\left(a\,k+c\,k^2-2\,k^2\,w+2\,\lambda\right)==0,\ w\right]$$

$$\left\{\left\{w\to\frac{a\,k+2\,\lambda}{2\,k^2}\right\}\right\}$$

We obtain the multiplier

```
Factor[Solve[w - a (-2 + γ)/(-2 + k γ) == 0 /. {w → (a k + c k^2 + 2 λ)/(2 k^2)}, λ]]
```

$$\left\{\left\{\lambda \to \frac{a\,k\,(2-4\,k+k\,\gamma)}{2\,(-2+k\,\gamma)}\right\}\right\}$$

We check the sign of $\lambda$:

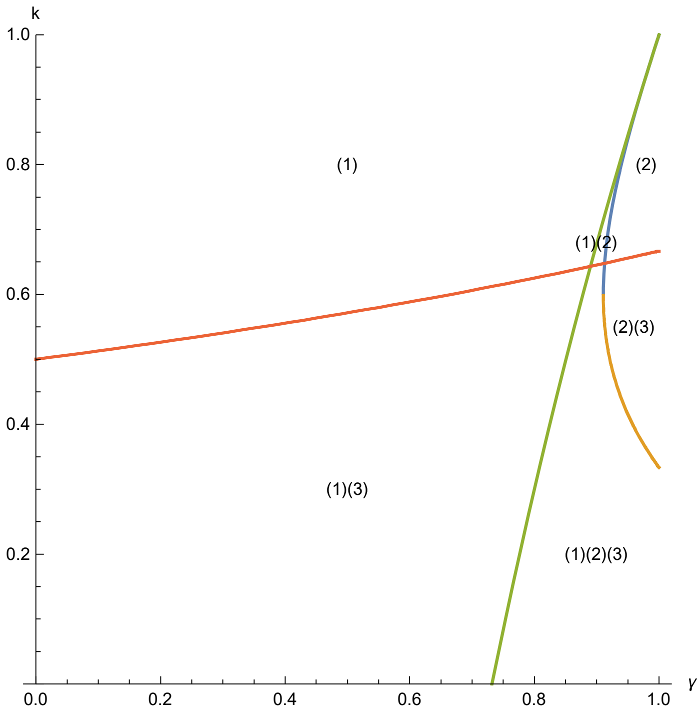
```
Simplify[Solve[-k (-2 a + 4 a k - 2 c k - a k γ + c k^2 γ)/(2 (-2 + k γ)) == 0, k]]
```

$$\left\{\{k \to 0\},\ \left\{k \to -\frac{2}{-4+\gamma}\right\}\right\}$$

The solution is interior if and only if $k < \frac{2}{4-\gamma}$.

We merge Cases 1, 2, and 3:

```
Plot[{-(2 + 3 γ - γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)),
  (-2 - 3 γ + γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)), 2 - 2/γ + γ, -2/(-4 + γ)}, {γ, 0, 1},
 PlotRange → {0, 1}, Epilog → {Text["(1)", {0.5, 0.8}], Text["(1)(3)", {0.5, 0.3}],
   Text["(1)(2)(3)", {0.9, 0.2}], Text["(2)", {0.98, 0.8}],
   Text["(2)(3)", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
 AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1]
```




### Comparison between the three

We find that the four regions have multiple local optimal solutions.
We need to compare which is the best between the local optimal solutions in each region.

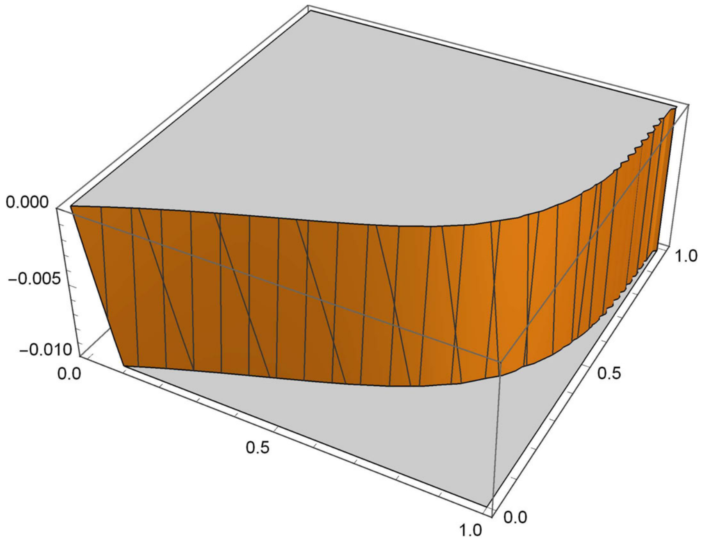
We compare the profits of U in Cases 1 and 3

```
Factor[-(a^2 (1 + k)^2 (-1 + γ) (2 + γ))/(4 (-2 + γ) (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)) - (a^2/8)]
```

$$-\frac{a^2\left(-8k+4\gamma+4k^2\gamma-2\gamma^2+2k\gamma^2-2k^2\gamma^2-\gamma^3+2k\gamma^3-k^2\gamma^3+\gamma^4+k^2\gamma^4\right)}{8(-2+\gamma)(1+\gamma)\left(-2-2k^2+2k\gamma+\gamma^2+k^2\gamma^2\right)}$$

We graphically show that the profit in Case 1 is higher

```
Plot3D[-(-8 k + 4 γ + 4 k^2 γ - 2 γ^2 + 2 k γ^2 - 2 k^2 γ^2 - γ^3 + 2 k γ^3 - k^2 γ^3 + γ^4 + k^2 γ^4)/(8 (-2 + γ) (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)),
 {γ, 0, 1}, {k, 0, 1}, PlotRange → {-0.01, 0}]
```




We explicitly derive the threshold value of k:

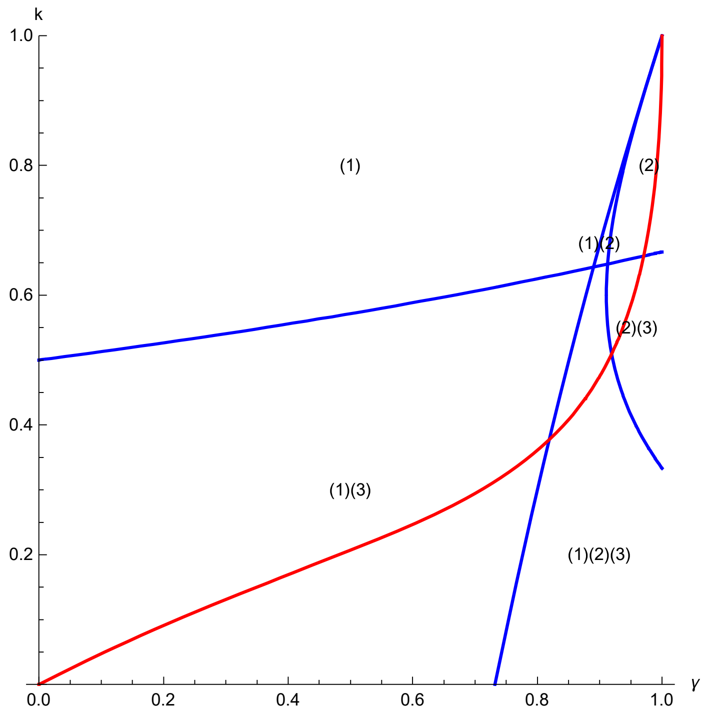
```
Simplify[
 Solve[-(-8 k + 4 γ + 4 k^2 γ - 2 γ^2 + 2 k γ^2 - 2 k^2 γ^2 - γ^3 + 2 k γ^3 - k^2 γ^3 + γ^4 + k^2 γ^4)/(8 (-2 + γ) (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)) == 0, k]]
```

$$\left\{\left\{k\to-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)}\right\},\right.$$

$$\left.\left\{k\to\frac{4-\gamma^2-\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)}\right\}\right\}$$

We merge the threshold value of k with the figure that summarizes threshold values of k in the three cases.

```
Plot[{-(2 + 3 γ - γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)),
  (-2 - 3 γ + γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)), 2 - 2/γ + γ, -2/(-4 + γ),
  -(-4 + γ^2 + γ^3 + Sqrt[-(2 + γ - γ^2)^2 (-4 + 4 γ - γ^2 + γ^4)])/(γ (4 - 2 γ - γ^2 + γ^3))}, {γ, 0, 1}, PlotRange → {0, 1},
 Epilog → {Text["(1)", {0.5, 0.8}], Text["(1)(3)", {0.5, 0.3}],
   Text["(1)(2)(3)", {0.9, 0.2}], Text["(2)", {0.98, 0.8}],
   Text["(2)(3)", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
 AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1,
 PlotStyle → {Blue, Blue, Blue, Blue, Red}]
```



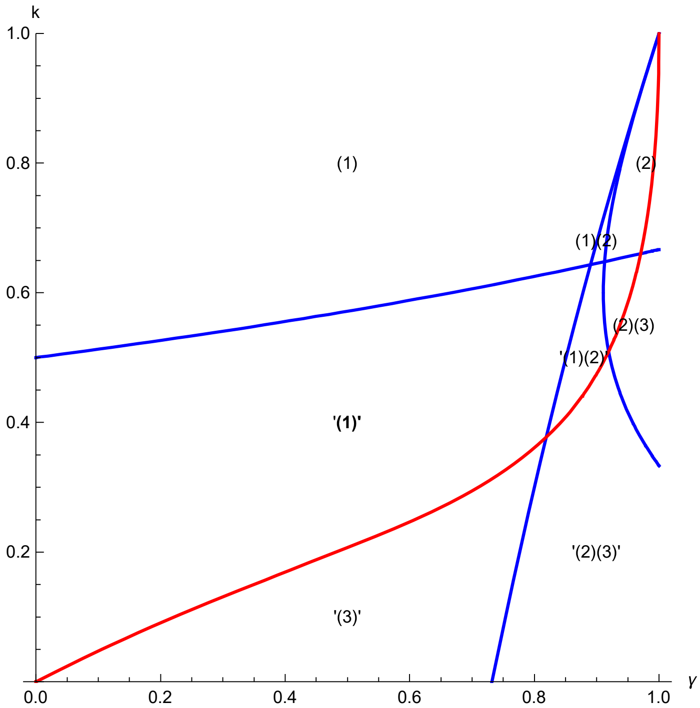

We use ‘( )‘ to express the updated areas:

$$\text{Plot}\left[\left\{-\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\right.\right.$$

$$\frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\ 2-\frac{2}{\gamma}+\gamma,\ -\frac{2}{-4+\gamma},$$

$$\left.-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)}\right\},\ \{\gamma,0,1\},$$

```
PlotRange → {0, 1}, Epilog → {Text["(1)", {0.5, 0.8}], Text["'(1)'", {0.5, 0.4}],
   Text["'(3)'", {0.5, 0.1}], Text["'(2)(3)'", {0.9, 0.2}],
   Text["'(1)(2)'", {0.88, 0.5}], Text["(2)", {0.98, 0.8}],
   Text["(2)(3)", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1,
PlotStyle → {Blue, Blue, Blue, Blue, Red}]
```




We check the crossing point:

$$\text{NSolve}\left[\frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)} ==\right.$$

$$\left.-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)},\ \gamma\right]$$

$\{\{\gamma \to 1.45463 + 2.00957\, i\}, \{\gamma \to 1.45463 - 2.00957\, i\},$
$\{\gamma \to -2.\}, \{\gamma \to 2.\}, \{\gamma \to -1.\}, \{\gamma \to 0.919166\}\}$

We update the figure:

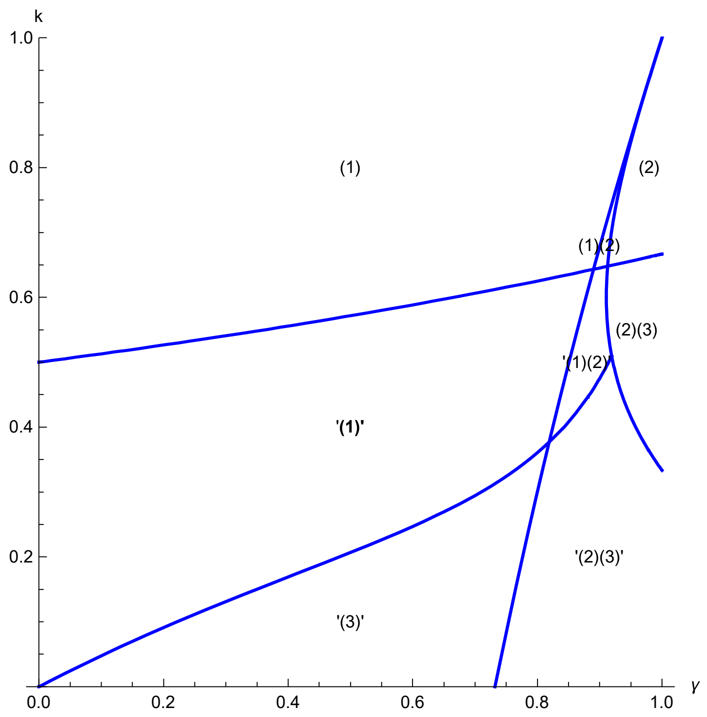
$$\texttt{Show}\Big[\texttt{Plot}\Big[\Big\{-\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},$$

$$\frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\ 2-\frac{2}{\gamma}+\gamma,\ -\frac{2}{-4+\gamma}\Big\},\ \{\gamma,0,1\},$$

```
  PlotRange → {0, 1}, Epilog → {Text["(1)", {0.5, 0.8}], Text["'(1)'", {0.5, 0.4}],
    Text["'(3)'", {0.5, 0.1}], Text["'(2)(3)'", {0.9, 0.2}],
    Text["'(1)(2)'", {0.88, 0.5}], Text["(2)", {0.98, 0.8}],
    Text["(2)(3)", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
  AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1,
  PlotStyle → {Blue, Blue, Blue, Blue, Red}],
```

$$\texttt{Plot}\Big[-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)},$$

```
  {γ, 0, 0.9191660940406087`}, PlotStyle → {Blue}]]
```



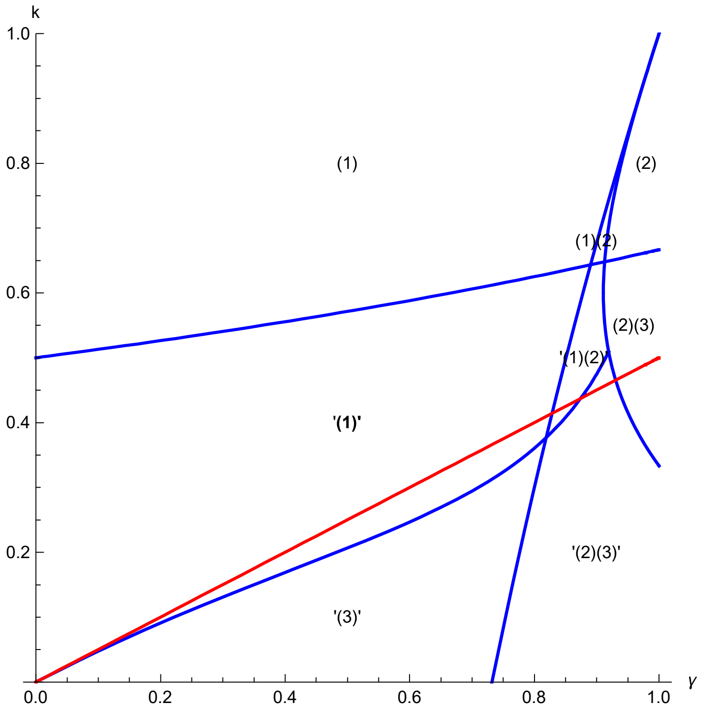

We compare the profits of U in Cases 2 and 3

$$\texttt{Factor}\left[\frac{a^2 k}{4\gamma}-\left(\frac{a^2}{8}\right)\right]$$

$$\frac{a^2\,(2k-\gamma)}{8\gamma}$$

We merge the threshold value of k with the latest updated figure:

```
Show[Plot[{-(2 + 3 γ - γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)),
   (-2 - 3 γ + γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)), 2 - 2/γ + γ, -2/(-4 + γ)}, {γ, 0, 1},
  PlotRange → {0, 1}, Epilog → {Text["(1)", {0.5, 0.8}], Text["'(1)'", {0.5, 0.4}],
    Text["'(3)'", {0.5, 0.1}], Text["'(2)(3)'", {0.9, 0.2}],
    Text["'(1)(2)'", {0.88, 0.5}], Text["(2)", {0.98, 0.8}],
    Text["(2)(3)", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
  AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1,
  PlotStyle → {Blue, Blue, Blue, Blue, Red}],
 Plot[-(-4 + γ^2 + γ^3 + Sqrt[-(2 + γ - γ^2)^2 (-4 + 4 γ - γ^2 + γ^4)])/(γ (4 - 2 γ - γ^2 + γ^3)), {γ, 0, 0.9191660940406087`},
  PlotStyle → {Blue}], Plot[γ/2, {γ, 0, 1}, PlotStyle → {Red}]]
```




We check the crossing points:

$$\text{NSolve}\left[\frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)} == \frac{\gamma}{2}, \gamma\right]$$

$$\text{NSolve}\left[-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)} == \frac{\gamma}{2}, \gamma\right]$$

$$\text{NSolve}\left[2-\frac{2}{\gamma}+\gamma == \frac{\gamma}{2}, \gamma\right]$$

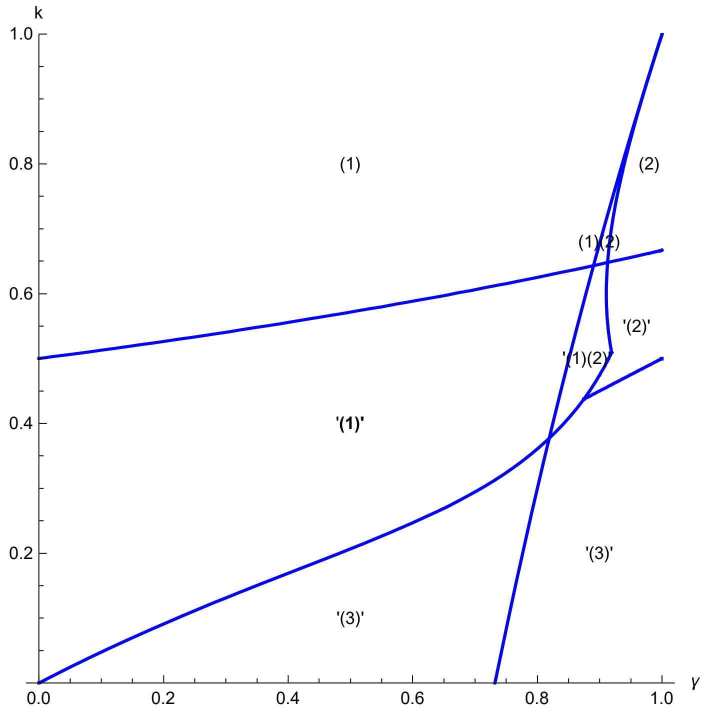
$\{\{\gamma \to 0.930548\}\}$

$\{\{\gamma \to -1.21634\}, \{\gamma \to 0.874327\}\}$

$\{\{\gamma \to -4.82843\}, \{\gamma \to 0.828427\}\}$

We update the figure:

```
Show[Plot[{-(2 + 3 γ - γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)), 2 - 2/γ + γ, -2/(-4 + γ)},
  {γ, 0, 1}, PlotRange → {0, 1}, Epilog →
   {Text["(1)", {0.5, 0.8}], Text["'(1)'", {0.5, 0.4}], Text["'(3)'", {0.5, 0.1}],
    Text["'(3)'", {0.9, 0.2}], Text["'(1)(2)'", {0.88, 0.5}], Text["(2)", {0.98, 0.8}],
    Text["'(2)'", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
  AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1,
  PlotStyle → {Blue, Blue, Blue, Blue, Red}],
 Plot[-(-4 + γ^2 + γ^3 + Sqrt[-(2 + γ - γ^2)^2 (-4 + 4 γ - γ^2 + γ^4)])/(γ (4 - 2 γ - γ^2 + γ^3)),
  {γ, 0, 0.9191660940406087`}, PlotStyle → {Blue}],
 Plot[(-2 - 3 γ + γ^2 + Sqrt[-28 + 4 γ + 37 γ^2 - 2 γ^3 - 7 γ^4])/(2 (-4 - γ + 2 γ^2)),
  {γ, 0, 0.9191660940406087`}, PlotStyle → {Blue}],
 Plot[γ/2, {γ, 0.8743271854931695`, 1}, PlotStyle → {Blue}]]
```




We compare the profits of U in Cases 1 and 2

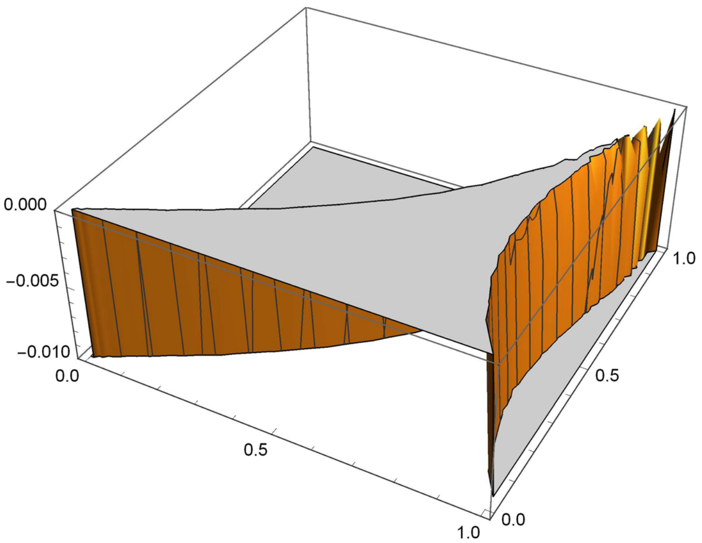
```
Factor[-(a^2 (1 + k)^2 (-1 + γ) (2 + γ))/(4 (-2 + γ) (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)) - (a^2 k)/(4 γ)]
```

$$-\frac{a^2 \left(-2 k+\gamma+k \gamma^2\right) \left(-2-2 k^2+\gamma+2 k \gamma-k^2 \gamma+\gamma^2+k^2 \gamma^2\right)}{4 (-2+\gamma) \gamma (1+\gamma) \left(-2-2 k^2+2 k \gamma+\gamma^2+k^2 \gamma^2\right)}$$

We graphically show that the profit in Case 1 is higher

```
Plot3D[-((-2 k + γ + k γ^2) (-2 - 2 k^2 + γ + 2 k γ - k^2 γ + γ^2 + k^2 γ^2))/(4 (-2 + γ) γ (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)),
 {γ, 0, 1}, {k, 0, 1}, PlotRange → {-0.01, 0}]
```

$$\text{Plot3D}\left[-\frac{\left(-2k+\gamma+k\gamma^2\right)\left(-2-2k^2+\gamma+2k\gamma-k^2\gamma+\gamma^2+k^2\gamma^2\right)}{4\,(-2+\gamma)\,\gamma\,(1+\gamma)\left(-2-2k^2+2k\gamma+\gamma^2+k^2\gamma^2\right)},\ \{\gamma,0,1\},\ \{k,0,1\},\ \text{PlotRange}\to\{-0.01,0\}\right]$$

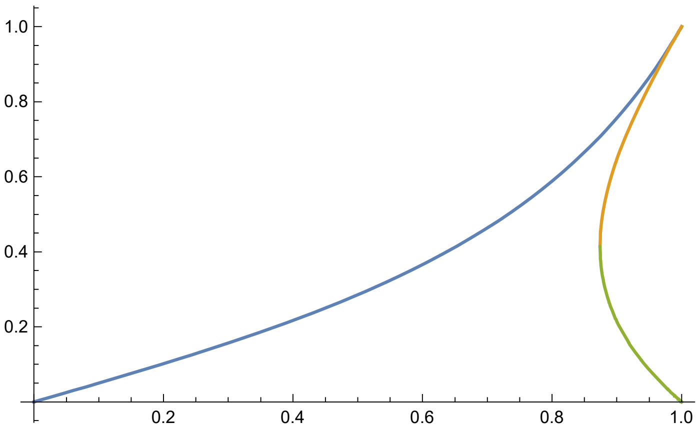



We explicitly derive the threshold values of k

$$\text{Solve}\left[-\frac{a^2\left(-2k+\gamma+k\gamma^2\right)\left(-2-2k^2+\gamma+2k\gamma-k^2\gamma+\gamma^2+k^2\gamma^2\right)}{4\,(-2+\gamma)\,\gamma\,(1+\gamma)\left(-2-2k^2+2k\gamma+\gamma^2+k^2\gamma^2\right)}==0,\ k\right]$$

$$\left\{\left\{k\to-\frac{\gamma}{-2+\gamma^2}\right\},\ \left\{k\to\frac{-\gamma-\sqrt{-4+6\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2}\right\},\ \left\{k\to\frac{-\gamma+\sqrt{-4+6\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2}\right\}\right\}$$

$$\text{Plot}\left[\left\{-\frac{\gamma}{-2+\gamma^2},\ \frac{-\gamma-\sqrt{-4+6\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2},\ \frac{-\gamma+\sqrt{-4+6\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2}\right\},\ \{\gamma,0,1\}\right]$$




We merge the threshold value of k with the latest updated figure:

$$\text{Show}\Big[\text{Plot}\Big[\Big\{-\frac{2+3\gamma-\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},\ 2-\frac{2}{\gamma}+\gamma,\ -\frac{2}{-4+\gamma}\Big\},$$

```
{γ, 0, 1}, PlotRange → {0, 1}, Epilog →
 {Text["(1)", {0.5, 0.8}], Text["'(1)'", {0.5, 0.4}], Text["'(3)'", {0.5, 0.1}],
  Text["'(3)'", {0.9, 0.2}], Text["'(1)(2)'", {0.88, 0.5}], Text["(2)", {0.98, 0.8}],
  Text["'(2)'", {0.96, 0.55}], Text["(1)(2)", {0.9, 0.68}]},
AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1,
PlotStyle → {Blue, Blue, Blue, Blue, Red}],
```

$$\text{Plot}\Big[-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\gamma-\gamma^2+\gamma^3\right)},$$

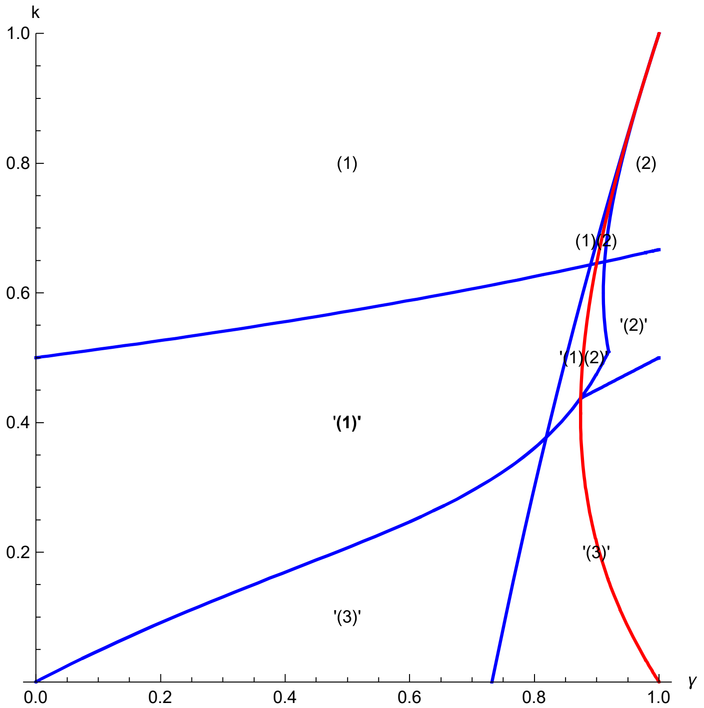
```
{γ, 0, 0.9191660940406087`}, PlotStyle → {Blue}],
```

$$\text{Plot}\Big[\frac{-2-3\gamma+\gamma^2+\sqrt{-28+4\gamma+37\gamma^2-2\gamma^3-7\gamma^4}}{2\left(-4-\gamma+2\gamma^2\right)},$$

```
{γ, 0, 0.9191660940406087`}, PlotStyle → {Blue}],
```

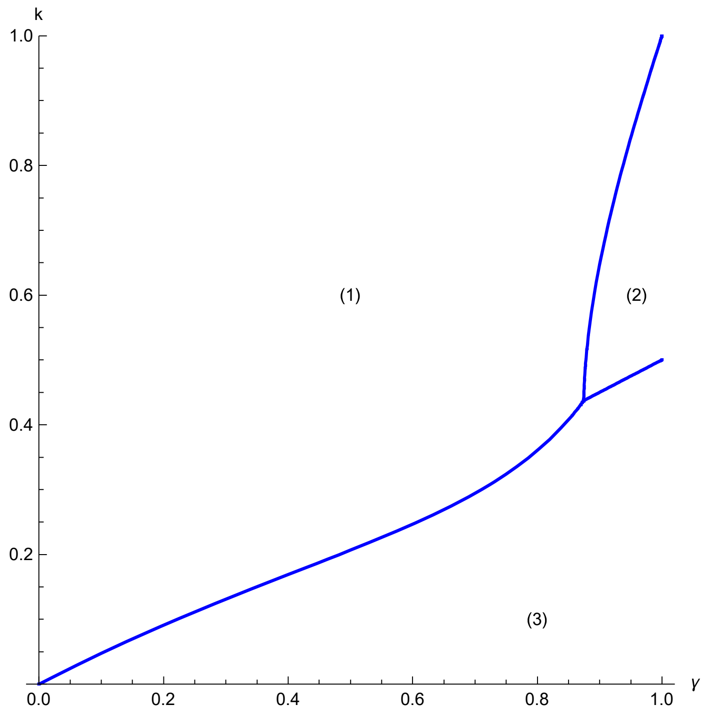
$$\text{Plot}\Big[\frac{\gamma}{2}, \{\gamma, 0.8743271854931695`, 1\}, \text{PlotStyle} \to \{\text{Blue}\}\Big],$$

$$\text{Plot}\Big[\Big\{\frac{-\gamma-\sqrt{-4+6\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2},\ \frac{-\gamma+\sqrt{-4+6\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2}\Big\}, \{\gamma, 0, 1\}, \text{PlotStyle} \to \{\text{Red}, \text{Red}\}\Big]\Big]$$




We update the figure:

```
Show[Plot[{-1}, {γ, 0, 1}, PlotRange → {0, 1},
  Epilog → {Text["(1)", {0.5, 0.6}], Text["(3)", {0.8, 0.1}], Text["(2)", {0.96, 0.6}]},
  AxesLabel → {"γ", "k"}, AxesOrigin → {0, 0}, AspectRatio → 1, PlotStyle → {Blue}],
 Plot[-(-4 + γ^2 + γ^3 + Sqrt[-(2 + γ - γ^2)^2 (-4 + 4 γ - γ^2 + γ^4)])/(γ (4 - 2 γ - γ^2 + γ^3)), {γ, 0, 0.8743271854931695`},
  PlotStyle → {Blue}], Plot[γ/2, {γ, 0.8743271854931695`, 1}, PlotStyle → {Blue}],
 Plot[{(-γ - Sqrt[-4 + 6 γ^2 - γ^4])/(-2 - γ + γ^2)}, {γ, 0.8743271854931695`, 1}, PlotStyle → {Blue}]]
```




*The above figure shows the realized case between the three cases for the parameter pair γ and k (c is assumed to be zero).*

## (iii) We check the condition that exclusion is attainable:

### Case 1

The profits of U and DI in Case 1

$$-\frac{a^2 (1+k)^2 (-1+\gamma)(2+\gamma)}{4(-2+\gamma)(1+\gamma)\left(-2-2k^2+2k\gamma+\gamma^2+k^2\gamma^2\right)}$$

$$-\frac{a^2 (-1+\gamma)\left(-2+2k-4k^2+3k\gamma-k^2\gamma+\gamma^2-k\gamma^2+2k^2\gamma^2\right)^2}{4(-2+\gamma)^2(1+\gamma)\left(-2-2k^2+2k\gamma+\gamma^2+k^2\gamma^2\right)^2}$$

The joint profits of U and DI in the exclusion case is $\frac{3}{16}(a-c)^2 = \frac{3a^2}{16}$.

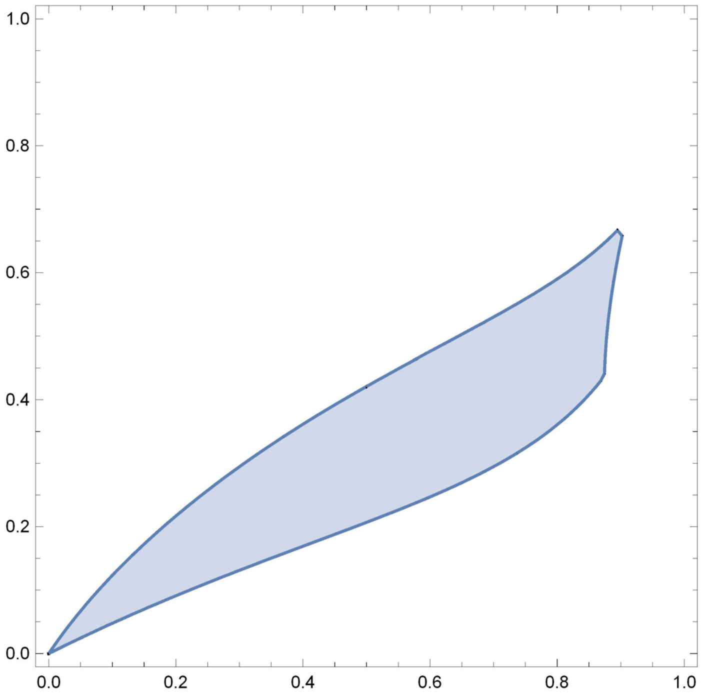
We compare the joint profits in the two cases:

$$\text{Factor}\left[\frac{3}{16}\ (a-c)^2-\left(-\frac{a^2\ (1+k)^2\ (-1+\gamma)\ (2+\gamma)}{4\ (-2+\gamma)\ (1+\gamma)\ \left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)}+\right.\right.$$
$$\left.\left.-\frac{a^2\ (-1+\gamma)\ \left(-2+2\,k-4\,k^2+3\,k\,\gamma-k^2\,\gamma+\gamma^2-k\,\gamma^2+2\,k^2\,\gamma^2\right)^2}{4\ (-2+\gamma)^2\ (1+\gamma)\ \left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)^2}\right)\right]$$

$$\left(a^2\ \left(-32\,k-48\,k^2-48\,k^4+48\,\gamma+16\,k\,\gamma+144\,k^2\,\gamma+48\,k^3\,\gamma+64\,k^4\,\gamma-44\,\gamma^2-64\,k\,\gamma^2-\right.\right.$$
$$28\,k^2\,\gamma^2-136\,k^3\,\gamma^2+32\,k^4\,\gamma^2-28\,\gamma^3+72\,k\,\gamma^3-52\,k^2\,\gamma^3+48\,k^3\,\gamma^3-56\,k^4\,\gamma^3+$$
$$40\,\gamma^4+8\,k\,\gamma^4+16\,k^2\,\gamma^4+48\,k^3\,\gamma^4+12\,k^4\,\gamma^4-4\,\gamma^5-36\,k\,\gamma^5+16\,k^2\,\gamma^5-44\,k^3\,\gamma^5+$$
$$\left.\left.8\,k^4\,\gamma^5-9\,\gamma^6+12\,k\,\gamma^6-18\,k^2\,\gamma^6+12\,k^3\,\gamma^6-9\,k^4\,\gamma^6+3\,\gamma^7+6\,k^2\,\gamma^7+3\,k^4\,\gamma^7\right)\right)\Big/$$
$$\left(16\ (-2+\gamma)^2\ (1+\gamma)\ \left(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\right)^2\right)$$

We draw the area in which exclusion is attainable in Case 1 by taking into account the area in which Case 1 is realized:

```
RegionPlot[
 {((-32 k - 48 k^2 - 48 k^4 + 48 γ + 16 k γ + 144 k^2 γ + 48 k^3 γ + 64 k^4 γ - 44 γ^2 - 64 k γ^2 - 28 k^2 γ^2 -
        136 k^3 γ^2 + 32 k^4 γ^2 - 28 γ^3 + 72 k γ^3 - 52 k^2 γ^3 + 48 k^3 γ^3 - 56 k^4 γ^3 + 40 γ^4 +
        8 k γ^4 + 16 k^2 γ^4 + 48 k^3 γ^4 + 12 k^4 γ^4 - 4 γ^5 - 36 k γ^5 + 16 k^2 γ^5 - 44 k^3 γ^5 +
        8 k^4 γ^5 - 9 γ^6 + 12 k γ^6 - 18 k^2 γ^6 + 12 k^3 γ^6 - 9 k^4 γ^6 + 3 γ^7 + 6 k^2 γ^7 + 3 k^4 γ^7)) /
     (16 (-2 + γ)^2 (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)^2) > 0 &&
    -(-4 + γ^2 + γ^3 + Sqrt[-(2 + γ - γ^2)^2 (-4 + 4 γ - γ^2 + γ^4)]) / (γ (4 - 2 γ - γ^2 + γ^3)) < k &&
    γ < 0.8743271854931695` ||
   ((-32 k - 48 k^2 - 48 k^4 + 48 γ + 16 k γ + 144 k^2 γ + 48 k^3 γ + 64 k^4 γ - 44 γ^2 - 64 k γ^2 -
        28 k^2 γ^2 - 136 k^3 γ^2 + 32 k^4 γ^2 - 28 γ^3 + 72 k γ^3 - 52 k^2 γ^3 + 48 k^3 γ^3 - 56 k^4 γ^3 +
        40 γ^4 + 8 k γ^4 + 16 k^2 γ^4 + 48 k^3 γ^4 + 12 k^4 γ^4 - 4 γ^5 - 36 k γ^5 + 16 k^2 γ^5 - 44 k^3 γ^5 +
        8 k^4 γ^5 - 9 γ^6 + 12 k γ^6 - 18 k^2 γ^6 + 12 k^3 γ^6 - 9 k^4 γ^6 + 3 γ^7 + 6 k^2 γ^7 + 3 k^4 γ^7)) /
     (16 (-2 + γ)^2 (1 + γ) (-2 - 2 k^2 + 2 k γ + γ^2 + k^2 γ^2)^2) > 0 &&
    (-γ - Sqrt[-4 + 6 γ^2 - γ^4]) / (-2 - γ + γ^2) < k && γ > 0.8743271854931695`}, {γ, 0, 1}, {k, 0, 1}]
```




## Case 2

The joint profits of U and DI in Case 2

$$\frac{a^2 k}{4 \gamma}$$

The joint profits of U and DI in the exclusion case is $\frac{3}{16}\ (a-c)^2 = \frac{3 a^2}{16}$.

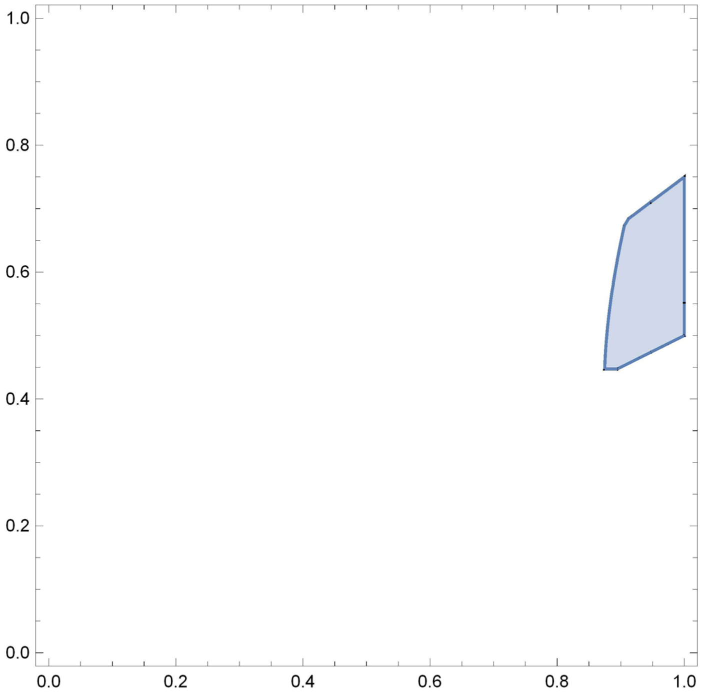
We compare the joint profits in the two cases:

$$\text{Factor}\left[\frac{3}{16}\ (a-c)^2 - \left(\frac{a^2\ k}{4\ \gamma}\right)\right]$$

$$-\frac{a^2\ (4\ k-3\ \gamma)}{16\ \gamma}$$

We draw the area in which exclusion is attainable in Case 2 by taking into account the area in which Case 1 is realized:

$$\text{RegionPlot}\left[-\frac{(4\ k-3\ \gamma)}{16\ \gamma} > 0\ \&\&\ \frac{\gamma}{2} < k < \frac{-\gamma-\sqrt{-4+6\ \gamma^2-\gamma^4}}{-2-\gamma+\gamma^2},\ \{\gamma,\ 0,\ 1\},\ \{k,\ 0,\ 1\}\right]$$




We merge the two regions:

**RegionPlot**

$$\text{RegionPlot}\Big[\Big\{\Big(\big(-32 k - 48 k^2 - 48 k^4 + 48 \gamma + 16 k \gamma + 144 k^2 \gamma + 48 k^3 \gamma + 64 k^4 \gamma - 44 \gamma^2 - 64 k \gamma^2 - 28 k^2 \gamma^2 - 136 k^3 \gamma^2 + 32 k^4 \gamma^2 - 28 \gamma^3 + 72 k \gamma^3 - 52 k^2 \gamma^3 + 48 k^3 \gamma^3 - 56 k^4 \gamma^3 + 40 \gamma^4 + 8 k \gamma^4 + 16 k^2 \gamma^4 + 48 k^3 \gamma^4 + 12 k^4 \gamma^4 - 4 \gamma^5 - 36 k \gamma^5 + 16 k^2 \gamma^5 - 44 k^3 \gamma^5 + 8 k^4 \gamma^5 - 9 \gamma^6 + 12 k \gamma^6 - 18 k^2 \gamma^6 + 12 k^3 \gamma^6 - 9 k^4 \gamma^6 + 3 \gamma^7 + 6 k^2 \gamma^7 + 3 k^4 \gamma^7\big)\big) \Big/ \left(16 (-2+\gamma)^2 (1+\gamma) \left(-2 - 2 k^2 + 2 k \gamma + \gamma^2 + k^2 \gamma^2\right)^2\right) > 0 \;\&\&$$

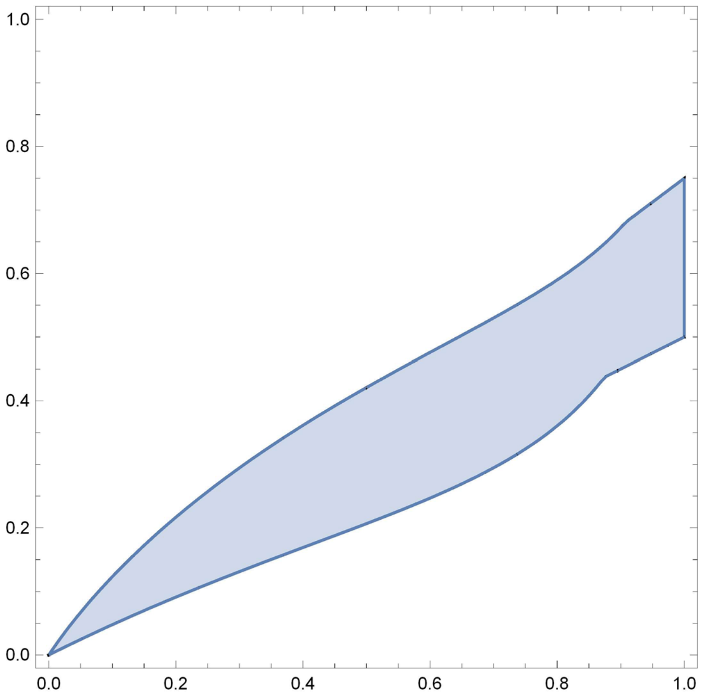
$$-\frac{-4 + \gamma^2 + \gamma^3 + \sqrt{-\left(2 + \gamma - \gamma^2\right)^2 \left(-4 + 4 \gamma - \gamma^2 + \gamma^4\right)}}{\gamma \left(4 - 2 \gamma - \gamma^2 + \gamma^3\right)} < k \;\&\&$$

$$\gamma < 0.8743271854931695\text{`} \;||$$

$$\Big(\big(-32 k - 48 k^2 - 48 k^4 + 48 \gamma + 16 k \gamma + 144 k^2 \gamma + 48 k^3 \gamma + 64 k^4 \gamma - 44 \gamma^2 - 64 k \gamma^2 - 28 k^2 \gamma^2 - 136 k^3 \gamma^2 + 32 k^4 \gamma^2 - 28 \gamma^3 + 72 k \gamma^3 - 52 k^2 \gamma^3 + 48 k^3 \gamma^3 - 56 k^4 \gamma^3 + 40 \gamma^4 + 8 k \gamma^4 + 16 k^2 \gamma^4 + 48 k^3 \gamma^4 + 12 k^4 \gamma^4 - 4 \gamma^5 - 36 k \gamma^5 + 16 k^2 \gamma^5 - 44 k^3 \gamma^5 + 8 k^4 \gamma^5 - 9 \gamma^6 + 12 k \gamma^6 - 18 k^2 \gamma^6 + 12 k^3 \gamma^6 - 9 k^4 \gamma^6 + 3 \gamma^7 + 6 k^2 \gamma^7 + 3 k^4 \gamma^7\big)\big) \Big/ \left(16 (-2+\gamma)^2 (1+\gamma) \left(-2 - 2 k^2 + 2 k \gamma + \gamma^2 + k^2 \gamma^2\right)^2\right) > 0 \;\&\&$$

$$\frac{-\gamma - \sqrt{-4 + 6 \gamma^2 - \gamma^4}}{-2 - \gamma + \gamma^2} < k \;\&\&\; \gamma > 0.8743271854931695\text{`} \;||$$

$$-\frac{(4 k - 3 \gamma)}{16 \gamma} > 0 \;\&\&\; \frac{\gamma}{2} < k < \frac{-\gamma - \sqrt{-4 + 6 \gamma^2 - \gamma^4}}{-2 - \gamma + \gamma^2}\Big\}, \{\gamma, 0, 1\}, \{k, 0, 1\}\Big]$$




**Case 3**

$$\frac{a^2}{8}$$

The joint profits of U and DI in the exclusion case is $\frac{3}{16}\ (a-c)^2 = \frac{3\,a^2}{16}$.

We compare the joint profits in the two cases:

```
Factor[3/16 (a - c)^2 - (a^2/8)]
```

$$\frac{a^2}{16}$$

We merge the three cases (horizontal axis: $\gamma$; vertical axis k).
The shaded area expresses the case in which exclusion is attainable.

$$\mathbf{RegionPlot}\Big[\Big\{\Big(\big(-32\,k-48\,k^2-48\,k^4+48\,\gamma+16\,k\,\gamma+144\,k^2\,\gamma+48\,k^3\,\gamma+64\,k^4\,\gamma-44\,\gamma^2-64\,k\,\gamma^2-28\,k^2\,\gamma^2-136\,k^3\,\gamma^2+32\,k^4\,\gamma^2-28\,\gamma^3+72\,k\,\gamma^3-52\,k^2\,\gamma^3+48\,k^3\,\gamma^3-56\,k^4\,\gamma^3+40\,\gamma^4+8\,k\,\gamma^4+16\,k^2\,\gamma^4+48\,k^3\,\gamma^4+12\,k^4\,\gamma^4-4\,\gamma^5-36\,k\,\gamma^5+16\,k^2\,\gamma^5-44\,k^3\,\gamma^5+8\,k^4\,\gamma^5-9\,\gamma^6+12\,k\,\gamma^6-18\,k^2\,\gamma^6+12\,k^3\,\gamma^6-9\,k^4\,\gamma^6+3\,\gamma^7+6\,k^2\,\gamma^7+3\,k^4\,\gamma^7\big)\Big)\Big/\Big(16\,(-2+\gamma)^2\,(1+\gamma)\,\big(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\big)^2\Big)>0\;\&\&$$

$$-\frac{-4+\gamma^2+\gamma^3+\sqrt{-\left(2+\gamma-\gamma^2\right)^2\left(-4+4\,\gamma-\gamma^2+\gamma^4\right)}}{\gamma\left(4-2\,\gamma-\gamma^2+\gamma^3\right)}<k\;\&\&$$

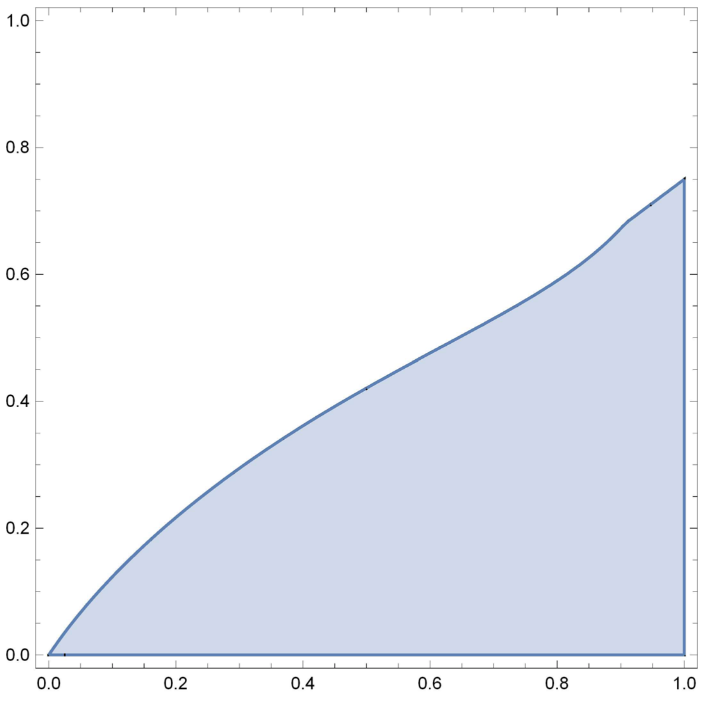
$$\gamma<0.8743271854931695\grave{}\;||$$

$$\Big(\big(-32\,k-48\,k^2-48\,k^4+48\,\gamma+16\,k\,\gamma+144\,k^2\,\gamma+48\,k^3\,\gamma+64\,k^4\,\gamma-44\,\gamma^2-64\,k\,\gamma^2-28\,k^2\,\gamma^2-136\,k^3\,\gamma^2+32\,k^4\,\gamma^2-28\,\gamma^3+72\,k\,\gamma^3-52\,k^2\,\gamma^3+48\,k^3\,\gamma^3-56\,k^4\,\gamma^3+40\,\gamma^4+8\,k\,\gamma^4+16\,k^2\,\gamma^4+48\,k^3\,\gamma^4+12\,k^4\,\gamma^4-4\,\gamma^5-36\,k\,\gamma^5+16\,k^2\,\gamma^5-44\,k^3\,\gamma^5+8\,k^4\,\gamma^5-9\,\gamma^6+12\,k\,\gamma^6-18\,k^2\,\gamma^6+12\,k^3\,\gamma^6-9\,k^4\,\gamma^6+3\,\gamma^7+6\,k^2\,\gamma^7+3\,k^4\,\gamma^7\big)\Big)\Big/\Big(16\,(-2+\gamma)^2\,(1+\gamma)\,\big(-2-2\,k^2+2\,k\,\gamma+\gamma^2+k^2\,\gamma^2\big)^2\Big)>0\;\&\&$$

$$\frac{-\gamma-\sqrt{-4+6\,\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2}<k\;\&\&\;\gamma>0.8743271854931695\grave{}\;||$$

$$-\frac{(4\,k-3\,\gamma)}{16\,\gamma}>0\;\&\&\;\frac{\gamma}{2}<k<\frac{-\gamma-\sqrt{-4+6\,\gamma^2-\gamma^4}}{-2-\gamma+\gamma^2}\;||$$

$$k<\frac{\gamma}{2}\Big\},\;\{\gamma,0,1\},\;\{k,0,1\}\Big]$$




*The above figure shows the parameter pair γ and k in which exclusion is attainable (c is assumed to be zero).*