Another relative performance measure

We change the payoff of firm $i$ ($i = 1, 2$) as follows:

$$U_i = \pi_i + \alpha \frac{\pi_i}{\pi_j} (i \neq j),$$

where $\pi_i$ is the profit of firm $i$ and $\alpha \in (0, a^2/4)$. $\alpha$ indicates the importance of relative performance for firm $i$'s management. Note that, when $\alpha = a^2/4$, the equilibrium price is zero in the competitive situation. We show it later.

First, we discuss joint-payoff maximization. The joint payoff is $\pi_1 + \pi_2 + \alpha (\pi_1/\pi_2 + \pi_2/\pi_1)$ and it is maximized when $y_1 = y_2 = a/2$. The resulting profit of each firm is $a^2/4$ (half of the monopoly profit), and the resulting payoff is:

$$U_1^C = \frac{a^2 + 8\alpha}{8},$$

where the superscript “$C$” denotes the outcome under the collusion.

Second, we discuss the deviation from the tacit collusion. Given the cooperative output of the rival, firm $2$, firm $1$ maximizes its payoff $U_1$. Given $y_2 = a/2$, the first-order condition is as below:

$$\frac{3a^2 + 16\alpha - 8ay_1}{4a} = 0.$$

From this, we obtain

$$y_1^D = \frac{3a^2 + 16\alpha}{8a}.$$
where the superscript "D" denotes the outcome when a firm deviates from the collusion. The resulting payoff is:

\[ U_1^D = \frac{(3a^2 + 16\alpha)^2}{8a}. \]

Third, we discuss the competitive situation. Each firm independently chooses its output so as to maximize its own payoff. We have the Cournot-Nash equilibrium as below:

\[ y_1^E = y_2^E = \frac{a + \sqrt{a^2 + 12\alpha}}{6}, \]

where the superscript "E" denotes the equilibrium outcome in the competitive phase. The resulting profit and payoff are given by:

\[ \pi_1^E = \pi_2^E = \frac{(a + \sqrt{a^2 + 12\alpha})(2a - \sqrt{a^2 + 12\alpha})}{18}, \quad U_1^E = U_2^E = \frac{a^2 + 6\alpha + a\sqrt{a^2 + 12\alpha}}{18}. \]

Given \( y_1^E \) and \( y_2^E \), the equilibrium price is

\[ p^E = \frac{2a - \sqrt{a^2 + 12\alpha}}{3}. \]

This is zero when \( \alpha = a^2/4. \)

**Results**  Given the collusive behavior of firm 2, firm 1 can increase its one-shot profit by deviating from the cartel. Its payoff is \( U_1^D \). This deviation induces the competition thereafter. Firm 1’s payoff at the competitive phase is \( U_1^E \). If firm 1 does not deviate from the collusion, its current payoff is \( U_1^C \). If firm 1 has no incentive for deviation now, it will have no incentive in future, as well. Thus, the tacit collusion is sustainable if and only if:

\[ \frac{U_1^C}{(1-\delta)} \geq U_1^D + \frac{\delta U_1^E}{1-\delta}. \]

Let \( \delta^* \) be the \( \delta \) satisfying the above equation with equality. The tacit collusion is sustainable if and only if \( \delta \geq \delta^* \). We have

\[ \delta^* = \frac{U_1^D - U_1^C}{U_1^D - U_1^E} = \frac{9(a^2 + 16\alpha)^2}{(7a^2 + 48\alpha)^2 - 32a^3\sqrt{a^2 + 12\alpha}}. \]

2
Following the tradition of this field, we measure the stability of collusion in terms of this minimum discount factor $\delta^*$. We have that an increase in $\alpha$ causes greater instability in collusive behavior.

**Proposition** $\delta^*$ is increasing in $\alpha$.

**Proof:** The partial derivative of $\delta^*$ with respect to $\alpha$ is

$$\frac{\partial \delta^*}{\partial \alpha} = \frac{576a^2(a^2 + 16\alpha)[2(7a^2 + 48\alpha)\sqrt{a^2 + 12\alpha} - a(13a^2 + 144\alpha)]}{\sqrt{a^2 + 12\alpha}((7a^2 + 48\alpha)^2 - 32a^3\sqrt{a^2 + 12\alpha})^2}.$$

If the term between the brackets in the numerator is positive, $\partial \delta^*/\partial \alpha$ is also positive. The following difference has the same sign with this term:

$$[2(7a^2 + 48\alpha)\sqrt{a^2 + 12\alpha}]^2 - [a(13a^2 + 144\alpha)]^2.$$

The difference is $27(a^2 + 16\alpha)^3$. This is positive. $\partial \delta^*/\partial \alpha$ is positive. Q.E.D.

## 2 Price competition

We consider the case in which the firms compete in price. We set the demand system in this case as follows:

$$q_1 = \begin{cases} 
0, & \text{if } a(1 - \gamma) + \gamma p_2 \leq p_1, \\
 a - p_1, & \text{if } p_1 \leq \frac{p_2 - a(1 - \gamma)}{\gamma}, \\
 \frac{a(1 - \gamma) - p_1 + \gamma p_2}{1 - \gamma^2}, & \text{otherwise}, 
\end{cases}$$

$$q_2 = \begin{cases} 
 a - p_2, & \text{if } a(1 - \gamma) + \gamma p_2 \leq p_1, \\
 0, & \text{if } p_1 \leq \frac{p_2 - a(1 - \gamma)}{\gamma}, \\
 \frac{a(1 - \gamma) - p_2 + \gamma p_1}{1 - \gamma^2}, & \text{otherwise}. 
\end{cases}$$

where $\gamma$ is a positive constant ($\gamma \in (0, 1)$). This demand system is related to standard demand functions for differentiated products. The payoff of firm $i$ ($i = 1, 2$) is given by
\( U_i = \pi_i - \alpha \pi_j \) (\( i \neq j \)), where \( \pi_i \) is the profit of firm \( i \) and \( \alpha \in (0, 1) \). \( \alpha \) indicates the importance of relative performance for firm \( i \)’s management.

First, we discuss joint-payoff maximization. The joint payoff is \((1 - \alpha)(\pi_1 + \pi_2)\) and it is maximized when \( p_1 = p_2 = a/2 \). The resulting profit of each firm is \( a^2/4(1 + \gamma) \) (half of the monopoly profit), and the resulting payoff is:

\[
U_1^C = \frac{(1 - \alpha)a^2}{4(1 + \gamma)},
\]

where the superscript “C” denotes the outcome under the collusion.

Second, we discuss the deviation from the tacit collusion. Given the cooperative output of the rival, firm 2, firm 1 maximizes its payoff \( U_1 \). When \( \alpha < (2 - 2\gamma - \gamma^2)/\gamma^2 \), the optimal deviation price is an interior solution. Given \( p_2 = a/2 \), the first-order condition is as below:

\[
\frac{(2 - (1 + \alpha)\gamma)a - 4p_1}{2(1 - \gamma)(1 + \gamma)} = 0.
\]

From this, we obtain

\[
p_D^1 = \frac{(2 - (1 + \alpha)\gamma)a}{4},
\]

where the superscript “D” denotes the outcome when a firm deviates from the collusion. The resulting payoff is:

\[
U_1^D = \frac{(4(1 - \alpha)(1 - \gamma) + (1 + \alpha)^2 \gamma^2)a^2}{16(1 - \gamma)(1 + \gamma)}.
\]

When \( \alpha \geq (2 - 2\gamma - \gamma^2)/\gamma^2 \), the optimal deviation price is a corner solution. This means that the demand for firm 2 is zero when firm 1 deviates from the collusion. Given \( p_2 = a/2 \), the optimal price is

\[
p_1 = \frac{(2\gamma - 1)a}{2\gamma}.
\]

The resulting payoff is:

\[
U_1^D = \frac{(2\gamma - 1)a^2}{4\gamma^2}.
\]
Third, we discuss the competitive situation. Each firm independently chooses its output so as to maximize its own payoff. We have the Bertrand-Nash equilibrium as below:

\[ p_1^E = p_2^E = \frac{(1 - \gamma) a}{2 - (1 - \alpha) \gamma}, \]

where the superscript “E” denotes the equilibrium outcome in the competitive phase. The resulting profit and payoff are given by:

\[ \pi_1^E = \pi_2^E = \frac{a^2(1 - \gamma)(1 + \alpha \gamma)}{(1 + \gamma)(2 - \gamma + \alpha \gamma)^2}, \quad U_1^E = U_2^E = \frac{a^2(1 - \alpha)(1 - \gamma)(1 + \alpha \gamma)}{(1 + \gamma)(2 - \gamma + \alpha \gamma)^2}. \]

**Results** Given the collusive behavior of firm 2, firm 1 can increase its one-shot profit by deviating from the cartel. Its payoff is \( U_1^D \). This deviation induces the competition thereafter. Firm 1’s payoff at the competitive phase is \( U_1^E \). If firm 1 does not deviate from the collusion, its current payoff is \( U_1^C \). If firm 1 has no incentive for deviation now, it will have no incentive in future, as well. Thus, the tacit collusion is sustainable if and only if:

\[
\frac{U_1^C}{(1 - \delta)} \geq U_1^D + \frac{\delta U_1^E}{1 - \delta}.
\]

Let \( \delta^* \) be the \( \delta \) satisfying the above equation with equality. The tacit collusion is sustainable if and only if \( \delta \geq \delta^* \). We have

\[
\delta^* = \frac{U_1^D - U_1^C}{U_1^D - U_1^E} = \begin{cases} 
\frac{(2 - (1 - \alpha) \gamma)^2}{4(2 - \alpha) - 8(1 - \alpha) \gamma + (1 - \alpha)^2 \gamma^2}, & \text{if } \alpha < (2 - 2\gamma - \gamma^2)/\gamma^2, \\
\frac{(2 - (1 - \alpha) \gamma)^2(\gamma^2 \alpha - 1 + \gamma + \gamma^2)}{-4 + 4(2 - \alpha) \gamma - (1 - 10\alpha + \alpha^2) \gamma^2 + (1 - \alpha) \gamma^3(2(1 + \alpha) \gamma - 3 - 5\alpha)}, & \text{if } \alpha \geq (2 - 2\gamma - \gamma^2)/\gamma^2.
\end{cases}
\]

Following the tradition of this field, we measure the stability of collusion in terms of this minimum discount factor \( \delta^* \). We have that an increase in \( \alpha \) causes greater instability in collusive behavior.
**Proposition** \( \delta^* \) is increasing in \( \alpha \).

**Proof:** The partial derivative of \( \delta^* \) with respect to \( \alpha \) is

\[
\frac{\partial \delta^*}{\partial \alpha} = \frac{U^D - U^C}{U^D - U^E}
\]

\[
= \begin{cases} 
4(1 - \gamma)(2 - (1 - \alpha)\gamma)(2 + (1 - \alpha)\gamma) & \text{if } \alpha < (2 - 2\gamma - \gamma^2)/\gamma^2, \\
\frac{[4(2 - \alpha) - 8(1 - \alpha)\gamma + (1 - \alpha)^2\gamma^2]^2}{(1 + \alpha)\gamma^4(2 - (1 - \alpha)\gamma)[2 - 5\gamma + 3\gamma^2 + 2\gamma^3 + 2(5\gamma - 3)\alpha + \gamma(-1 + 5\gamma - 2\gamma^2)\alpha^2]} & \text{if } \alpha \geq (2 - 2\gamma - \gamma^2)/\gamma^2.
\end{cases}
\]

In the former case, we easily find that this is positive \((\gamma \in (0, 1) \text{ and } \alpha(0, 1))\). In the latter case, note that \((2 - 2\gamma - \gamma^2)/\gamma^2\) is larger than 1 if \(\gamma \leq 3/5\). This means that the latter case appears only if \(\gamma > 3/5 \text{ because } \alpha \leq 1\). Now consider the term between the brackets in the numerator. We can easily show that \(2 - 5\gamma + 3\gamma^2 + 2\gamma^3\) and the coefficients of \(\alpha\) and \(\alpha^2\) are positive for any \(\gamma > 3/5\). This means that the term between the brackets is positive. Therefore, in the former and the latter cases, the partial derivative of \(\delta^* \) is positive. Q.E.D.