Supplementary material

Proof of the model in Section 3.2 We now prove the result. To prove the result, we have to consider six deviation patterns: (i) the candidate locating at y relocates at $x < y$; (ii) the candidate locating at y relocates at $x \in (y, 1-y)$; (iii) the candidate locating at y relocates at $x \geq 1-y$; (iv) the candidate locating at $1-y$ relocates at $x \leq y$; (v) the candidate locating at $1-y$ relocates at $x \in (y, 1-y)$; and (vi) the candidate locating at $1-y$ relocates at $x > 1-y$. We now discuss the six deviation patterns.

In the first case, the locations of the candidates are depicted in Figure 1A. The shaded circle (c_1) is the relocating candidate.

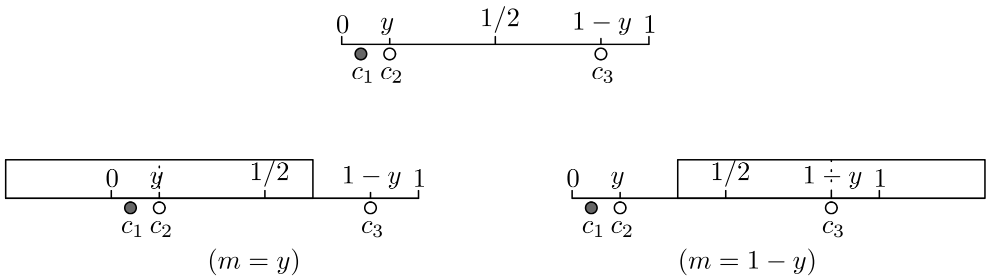

Figure 1A: Case 1 ($x < y$)

First, we suppose that $m = y$. In the first round, the shares of the candidates (c_1, c_2, and c_3) are $1/2 - (y-x)/2$, $(y-x)/2 + 1/2 - y$, and y. c_2, who locates at y, takes a share higher than c_3, who locates at $1-y$, and then wins in the first round if $y < 1/4$. In the second round, c_2 takes a share higher than $1/2$, that is, c_1 loses. Second, we suppose that $m = 1-y$. In the first round, the shares of c_1, c_2, and c_3 are 0, y, and $1-y$, respectively. c_1 loses in the first round. Therefore, c_1 (the deviating candidate) wins with probability 0, and the deviation is not beneficial.

In the second case, the locations of the candidates are depicted in Figure 2A. The shaded circle (c_2) is the relocating candidate.

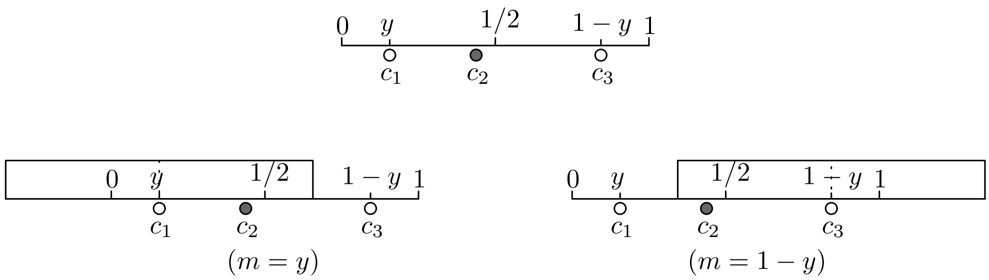

Figure 2A: Case 2 ($x \in (y, 1-y)$)

First, we suppose that $m = y$. In the first round, the shares of the candidates (c_1, c_2, and c_3) are $1/2 + (x-y)/2$, $(x-y)/2 + (1-x-y)/2$, and $(3y-x)/2$, respectively, if $x < 3y$, and $1/2 + (x-y)/2$,

$1/2 - (x - y)/2$, and 0 otherwise. c_1, who locates at y, takes a share higher than $1/2$. In the second round, c_1 wins, that is, c_2 loses. Second, we suppose that $m = 1 - y$. The inverse result holds, that is, c_3 wins in the second round, and c_2 loses. Therefore, the deviating candidate, c_2, wins with probability 0, and the deviation is not beneficial.

In the third case, the locations of the candidates are depicted in Figure 3A. The shaded circle (c_3) is the relocating candidate.

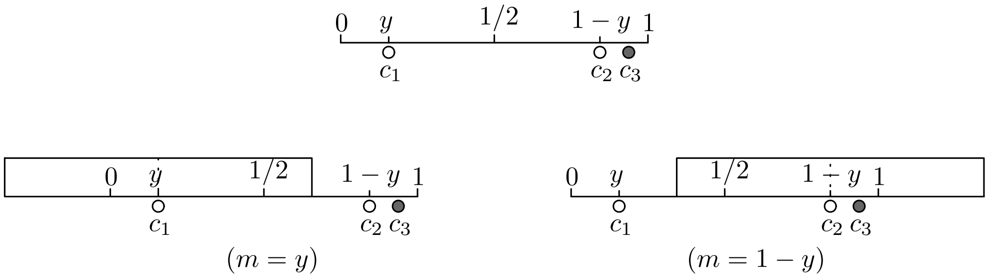

Figure 3A: Case 3 ($x > 1 - y$)

The locations are similar to those in the first case. Therefore, the deviating candidate wins with probability 0, and the deviation is not beneficial.

In the fourth case, the locations of the candidates are depicted in Figure 4A. The shaded circle (c_1) is the relocating candidate.

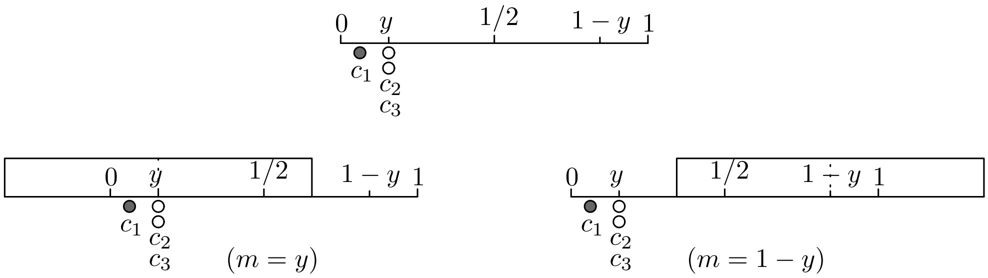

Figure 4A: Case 4 ($x < y$)

Suppose that the deviating candidate, c_1, wins in the first round. Under both cases ($m = y$ and $m = 1 - y$), the rival candidate in the second round locates at y. The rival (c_2 or c_3) can take a share of more than half and then wins in the second round. Therefore, the deviating candidate wins with probability 0, and the deviation is not beneficial.

In the fifth case, the locations of the candidates are depicted in Figure 5A. The shaded circle (c_3) is the relocating candidate.

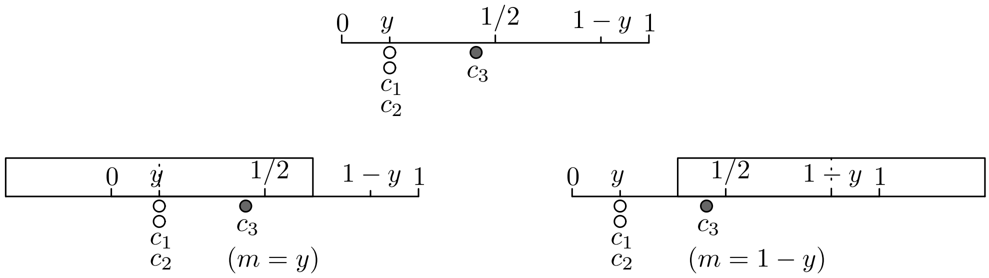

Figure 5A: Case 5 ($x \in (y, 1-y)$)

First, we suppose that $m = y$. We also suppose that the deviating candidate, c_3, wins in the first round. The rival candidate in the second round locates at y. The rival (c_1 or c_2) takes a share higher than 1/2. In the second round, c_3 loses. Second, we suppose that $m = 1-y$. The deviating candidate, c_3, takes its share higher than 1/2 and then wins in both rounds. Therefore, c_3 wins with probability 1/2. The deviation does not improve the probability of winning the election.

In the sixth case, the locations of the candidates are depicted in Figure 6A. The shaded circle (c_3) is the relocating candidate.

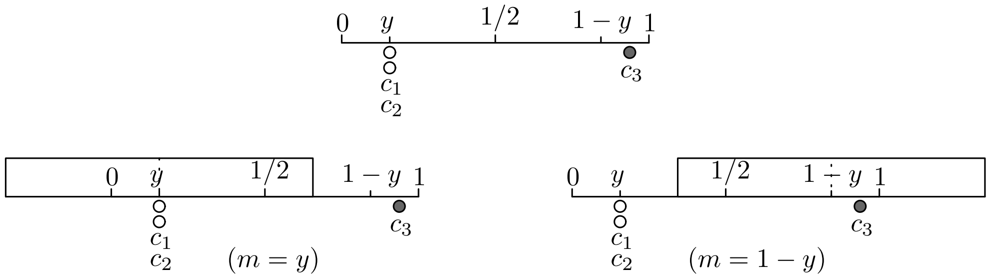

Figure 6A: Case 6 ($x > 1-y$)

First, we suppose that $m = y$. We also suppose that the deviating candidate, c_3, wins in the first round. The rival candidate in the second round locates at y. The rival (c_1 or c_2) takes a share higher than 1/2. In the second round, c_3 loses. Therefore, at best, c_3 wins with probability 1/2. The deviation does not improve the probability of winning the election.

From the six cases, we find that the result in Section 3.2 holds. Q.E.D.

Proof of the model in Section 3.2 (without the runoff system) We first show that the location pattern in Figure 2 is not an equilibrium outcome in a one-round election.

First, we suppose that $m = y$. The shares of the candidates are $(1-y)/2$, $(1-y)/2$, and y. $(1-y)/2 > y$ if and only if $y < 1/3$. We have assumed $y \in (0, 1/4)$. Therefore, one of the left-hand-side candidates wins. Second, we suppose that $m = 1-y$. The shares of the candidates are $y/2$, $y/2$, and $1-y$. $y/2 < 1-y$ if and only if $y < 2/3$. We have assumed $y \in (0, 1/4)$. Therefore, the right-hand-side candidate wins. The probabilities of winning the election are 1/4, 1/4, and 1/2.

Suppose that one of the left-hand-side candidates relocates at $x(< y)$. We suppose that $m = y$. The shares of the candidates are $1/2-(y-x)/2$, $1/2-y+(y-x)/2$, and y. If x is sufficiently close to y, the share of the relocating candidate is close to $1/2$. We have assumed $y \in (0, 1/4)$. Therefore, the relocating candidate wins. The probability is $1/2$. We conclude that the location pattern in Figure 2 is not an equilibrium outcome when the runoff system does not exist.

We now show that pure strategy equilibria do not exist in a one-round election. We consider three location patterns: (1) three candidate locate at one point; (2) two candidates locate at one point, and the other candidate locates elsewhere; (3) each candidate locates at a different point.

First, we consider the following three cases, in which three candidates locate at one point (see Figure B1).

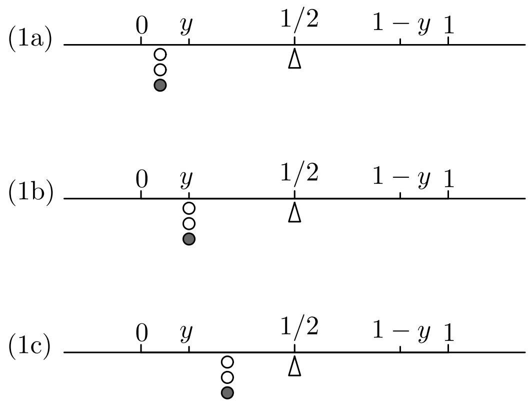

Figure B1: Three candidates locate at one point

In cases 1a, 1b, and 1c, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is at least $1/2$ (it wins when $m = 1 - y$), that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location patterns do not appear in equilibrium.

Second, we consider the following three cases, in which two candidates locate at one point the other candidate locates elsewhere (see Figure B2).

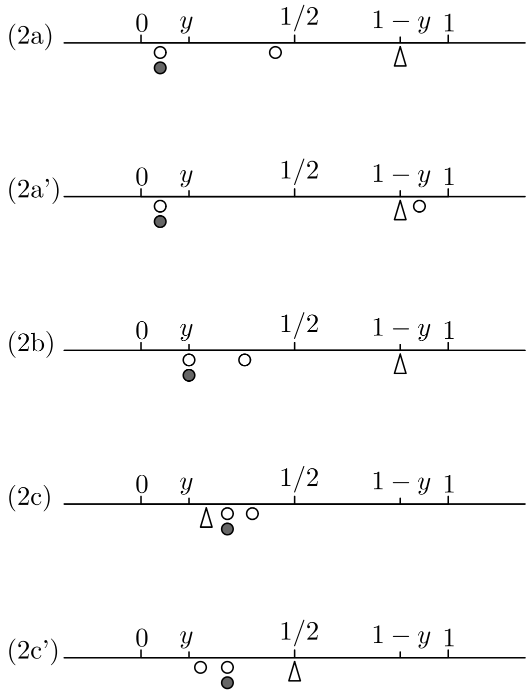

Figure B2: Two candidates locate at one point

In case 2a (2a'), two candidates locate at point x, which satisfies $x < y$. In this case, the other candidate should locate at x', which satisfies $x < x' < (1 + 6y - 3x)/3$ because its probability of winning is 1 due to the location (note that, if $9y > 2 + 3x$, $(1 + 6y - 3x)/3$ is larger than $1 - y$; see case 2a'). Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle $(1 - y)$, the probability of winning the election is at least 1/2, that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

In case 2b, two candidates locate at point y. In this case, the other candidate should locate at x', which satisfies $y < x' < (1 + 3y)/3$ because its probability of winning is 1 due to the location. Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is at least 1/2, that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

In case 2c (2c'), two candidates locate at point x, which satisfies $y < x \leq 1/2$. If $x < y + 1/6$, the other candidate should locate at x', which satisfies $x < x' < (1 + 6y - 3x)/3$ because its probability of winning is 1 by the location (see case 2c); otherwise, it should locate at point x'', which induces its probability of winning to be 1/2 (in case 2c, it wins when $m = 1 - y$; in case 2c', it wins when $m = y$). Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is 1/2; that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

Finally, we consider the following five cases, in which each candidate locates at a different point (see Figure B3).

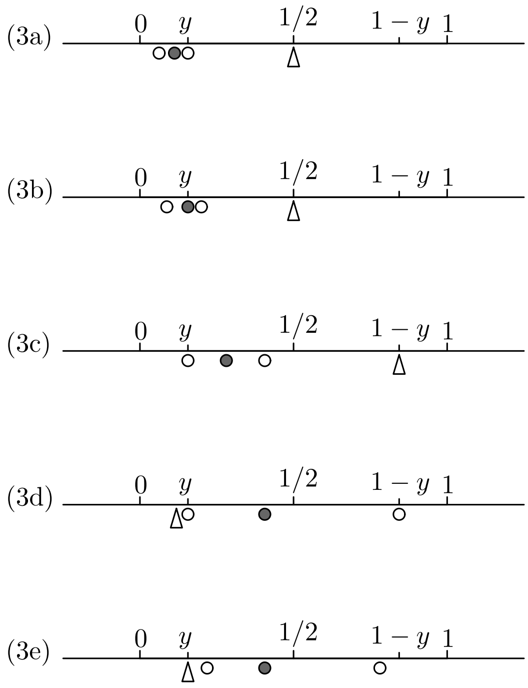

Figure B3: Each candidate locate at a different point

In case 3a, two candidates locate at points x_i, which satisfies $x_i < y$ $(i = 1, 2)$. In this case, the other candidate should locate at x', which satisfies $x' < 4y - x_1 - 2x_2$, because its probability of winning is 1 due to the location. Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is at least 1/2; that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

In case 3b, one candidate locates at point x, which satisfies $x < y$, and another candidate locates at point y. In this case, the other candidate should locate at x', which satisfies $x' < 2y - x_1$, because its probability of winning is 1 due to the location. Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is at least 1/2; that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

In case 3c, one candidate locates at point y, and the other two candidates locate at points x_i, which satisfies $y < x_i < 1 - y$. The candidate locating at point y wins with probability 1/2 (it wins when $m = y$). The right-hand-side candidate also wins with probability $1/2$ (it wins when $m = 1-y$). Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is $1/2$; that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

In case 3d, two candidates locate at points y and $1-y$, and the other candidate locates at points x, which satisfies $y < x < 1-y$. The candidate locating at point y wins with probability $1/2$ (it wins when $m = y$). The right-hand-side candidate also wins with probability $1/2$ (it wins when $m = 1-y$). Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is $1/2$; that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

In case 3e, three candidates locate at points x_i, which satisfies $y < x_i < 1-y$ $(i = 1, 2, 3)$. This case is similar to case 3d. Given the locations of the three candidates, if the shaded candidate relocates at the point indicated by the triangle, the probability of winning the election is $1/2$; that is, the relocation enhances the winning probability of the shaded candidate. Therefore, the location pattern does not appear in equilibrium.

From the discussion, we find that stable location patterns do not exist in the setting, that is, pure strategy equilibria do not exist in a one-round election. Q.E.D.